Generalized Nash equilibrium seeking for directed nonsmooth multi-cluster games via a distributed Lipschitz algorithm

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***Abstract*—This paper investigates a generalized Nash equilibrium (GNE) seeking strategy for a class of nonsmooth multi-cluster games. Each cluster consists of several players. The inter-cluster graph is directed and weight-unbalanced. Moreover, in contrast to previous works of multi-cluster games, coupled nonsmooth inequality constraints, resource allocation constraints, and nonsmooth payoff functions are considered simultaneously in these multi-cluster games. For seeking the GNE of these games, a distributed Lipschitz algorithm with the proximal-splitting scheme is proposed. Then convergence analysis of this designed algorithm is deduced based on Lyapunov stability theory and convex optimization theory. Finally, some simulation results are provided in this paper, which show the efficacy of the distributed GNE seeking algorithm.**

***Index Terms*—Distributed algorithms, Proximal operator, Multi-cluster games, Nonsmooth functions, Distributed GNE seeking**

I. Introduction

In this paper, a Lipschitz GNE seeking strategy for a class of nonsmooth multi-cluster games is investigated here. Each cluster consists of several players. In order to minimize its payoff function, each player in this game employs the local strategy depending on its own and other players' decisions. This kind of noncooperative games has gained significant attention in cyber security, social networks, and smart grids, etc [1]–[5]. Each leader of the clusters can exchange the information through a directed graph. With the requirement of the communication topology, every player will only receive information from its neighboring players. Consequently, distributed Nash equilibrium seeking strategy is a critical problem for this noncooperative games [6]–[9]. This multi-cluster game problem considers both nonsmooth payoff functions and nonsmooth inequality constraint. The nonsmooth functions may cause vibration of the state of the system. There are few works can deal with this problem with coupled nonsmooth constraints via a Lipschitz-continuous algorithm. Moreover, this problem contains two coupled constraints. One is the inter-cluster resource allocation condition. Another one is the inner-cluster nonsmooth inequality constraint. Then another main difficulty of this problem is to decouple these constraints and design the corresponding algorithm in a fully distributed way. Due to important applications and challenges mentioned above, these problems have attracted increasing attention.

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A. Literature review

1) Multi-Cluster Games: In many important problems, players' competition and cooperation behaviours exist. These problems can be modeled as multi-cluster games that combine noncooperative games and distributed optimization. In multi-cluster games, leaders of clusters participate in an inter-cluster noncooperative game. A group of numerous local players are included in a cluster. They can be regarded as inner-cluster collaborators that optimize the cluster payoff function cooperatively. In recent years, distributed strategies for GNE seeking of multi-cluster games have been widely investigated. For a class of zero-sum games with two sub-networks, [10] investigated distributed NE seeking algorithms. [11] proposed a unified strategy for GNE seeking of multi-cluster games. This method reduced the computation cost and the communication cost. For multi-cluster games, [12] designed an extremum seeker, where agents do not have accurate expressions of local cost functions. Aiming to multi-cluster games with inequality constraints, [13] developed an average consensus approach and a finite-time distributed algorithm for GNE seeking. [14] proposed a distributed projected GNE seeking algorithm through gradient descent for multi-cluster games, where the global inequality constraints, local inequality constraints, and local convex set constraints exist simultaneously.

2) Communication Networks: Communication networks among players play a key role for designing and analysing distributed GNE seeking algorithms. Many existing GNE seeking algorithms with distributed methods are designed for multi-cluster games and noncooperative games based on the assumption of undirected graphs. However, directed graphs have broader applications than that of undirected graphs in noncooperative games and multi-cluster games. Furthermore, it is noteworthy that an undirected-graph-based algorithm may fail to converge under directed graphs. As a result, it is worth studying distributed GNE seeking algorithms under directed graphs. [15] investigated two distributed variational algorithms for GNE seeking of weight-balanced aggregative games with differentiable and nonsmooth cost functions, respectively. [16] proposed an NE seeking algorithm for weight-balanced aggregative games with a heterogeneous second-order nonlinear multi-agent system. [17] developed a NE seeking strategy with discrete-time gradient-free algorithm for multi-cluster games with inner-cluster digraphs. The adjacent matrices of the digraphs in these problems are doubly-stochastic. [18] designed a NE seeking algorithm of strongly connected consistency-constrained multi-cluster games. For multi-cluster games with jointly connected switching digraphs, [19] proposed a distributed consensus-based NE seeking algorithm. [20] proposed continuous-time GNE seeking algorithms for weight-balanced and weight-unbalanced digraphs.

3) Nonsmooth Analysis: As a natural characteristic, nonsmoothness often arises in optimization and game problems in real-world engineering areas. The subgradient-based algorithm is inherently developed [21]–[25], whose convergence was deduced according to the nonsmooth analysis. [22] proposed a distributed hybrid algorithm for constrained nonsmooth optimization, including a continuous-time differential inclusion mapping and a discrete-time jump set triggered mapping. [23] developed a distributed subgradient-based GNE seeking algorithm for nonsmooth set-constrained multi-cluster games with additional coupled nonsmooth inequality constraints. [25] investigated a dynamic average consensus-based GNE seeking algorithm for nonsmooth coupled constrained and heterogeneous local constrained aggregative games. Hoever, since subgradients are non-continues, these algorithms may face challenges in convergence analysis and may not be easily applicable in real-world systems. As a comparison, distributed Lipschitz algorithms based on the proximal operator have been investigated [26]–[30], as it can be easily analysed with the Lyapunov stability theory. In order to solve a class of composite nonsmooth consensus optimization problems, [27] investigated a distributed double proximal based primal-dual algorithm. For solving nonsmooth consensus convex optimizations with resource allocation constraints, [28] designed distributed proximal-gradient algorithms with derivative feedback for second-order multi-agent systems. For nonsmooth mixed-order multi-cluster games, [30] developed a distributed proximal-gradient NE seeking algorithm.

B. Contribution

Motivated by the above challenges and limitation of the previous works, a GNE seeking strategy for nonsmooth multi-cluster games is investigated here. Each cluster consists of several players. The inter-cluster graph is directed and weight-unbalanced. Moreover, in contrast to previous works of multi-cluster games, coupled nonsmooth inequality constraints, resource allocation constraint, and nonsmooth payoff functions are considered in this multi-cluster game simultaneously. A distributed proximal-based Lipschitz algorithm is designed for this class of nonsmooth multi-cluster games with directed inter-cluster graph and coupled nonsmooth inequality constraints. Contributions of this work are listed as below.

(i) This paper aims to solve a class of nonsmooth multi-cluster games with inter-cluster directed graph and inner-cluster coupled nonsmooth inequality constraints. In contrast to [21]–[25], a distributed Lipschitz algorithm is designed for nonsmooth multi-cluster games. Compared with [19], [26]–[30], this paper considers inter-cluster weight-unbalanced directed graphs and nonsmooth constraints for nonsmooth multi-cluster games. As the result, a main difficulty of this problem is to decouple these constraints and design the corresponding algorithm in a fully distributed way simultaneously.

(ii) In this work, a distributed Lipschitz algorithm employing proximal operators is developed. A novel proximal splitting scheme is employed for making composite nonsmooth Lagrangian functions proximable, and therefore guarantee Lipschitz continuity of the proposed algorithm. This algorithm can deal with the nonsmooth constraint coupled with the Lagrangian variable.

(iii) In this work, the convergence analysis of the proposed algorithm is conducted. By combining convex optimization theory with Lyapunov stability theory to decouple nonsmooth functions and Lagrangian multipliers, it offers a novel approach to analyse of nonsmooth multi-cluster games.

This paper is scheduled as the following parts. Section II shows basic mathematical preliminaries for graph theory and the proximal operator. In Section III, the distributed nonsmooth multi-cluster game under inter-cluster directed graph and nonsmooth inequality constraints is presented. In Section IV, a distributed primal-dual Lipschitz algorithm with a proximal splitting scheme is proposed. The convergence analysis of this proposed algorithm is also deduced in Section IV. In Section V, simulation results for a multi-cluster game with sixteen players are provided to illustrate the effectiveness of the proposed algorithm. Finally, Section VI presents the conclusion and future extensions for this work.

II. Mathematical Preliminaries

A. Graph Theory

Let $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ denote a weighted graph $\mathcal{G}$, in which $\mathcal{V}=\{1,\dots,n\}$ and $\mathcal{E}$ means the set of nodes and edges respectively. The weighted adjacency matrix is denoted as $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{n\times n}$. An edge in a undirected graph is denoted as $e_{ij}\in\mathcal{E}$, which means that agents i and j can exchange message from each other. $e_{ij}\in\mathcal{E}$ means that $a_{ij}=a_{ji}>0$, or $a_{ij}=0$, and $a_{ii}=0, i\in\mathcal{I}$. Moreover, an edge $e_{ij}\in\mathcal{E}$ in a directed graph $\mathcal{G}$ means that agent i can receive message from agent j. $e_{ij}\in\mathcal{E}$ means that $a_{ij}>0$, and $a_{ii}=0, i\in\mathcal{I}$. In a directed graph, the in-degree of agent i can be represented as $d_i^{in}=\sum_{j=1}^{n}a_{ij}$, and the out-degree of agent i can be represented as $d_i^{out}=\sum_{j=1}^{n}a_{ji}$.

A neighbour of agent i being agent j can be shown as $j\in\mathcal{N}_i$. The set of real numbers is denoted as $\mathbb{R}$. L_n denotes a Laplacian matrix, where $L_n=D-\mathcal{A}$. $D\in\mathbb{R}^{n\times n}$ is diagonal matrix, where for each $i\in\{1,\dots,n\}$, $D_{(i),(i)}=\sum_{j=1}^{n}a_{ij}$. The Kronecker product of L_n and I_q can be shown as $\mathbf{L}\triangleq L_n\otimes I_q\in\mathbb{R}^{nq\times nq}$, where I_q is the q-dimensional identity matrix. The Euclidean norm of a vector $a\in\mathbb{R}^n$ is shown as $\|a\|$. The l_1 norm of a vector $a\in\mathbb{R}^n$ is shown as $\|a\|_1$. The set of positive real numbers is denoted as $\mathbb{R}^+$. The diagonal matrix is denoted as $\mathrm{diag}\{b_1, \cdots, b_n\} \in \mathbb{R}^{n\times n}$. For this matrix, the i-th diagonal element is represented like $b_i \in \mathbb{R}$ for any $i \in \{1, \cdots, n\}$. The n-dimensional null matrix is denoted as O_n. The vector of zeros with n-dimension is denoted as $\mathbf{0}_n \in \mathbb{R}^n$. $(\cdot)^T$ denotes transpose of matrix.

For the zero eigenvalue of the Laplacian matrix L_n, the corresponding left eigenvector is denoted as a positive vector $h = (h_1, h_2, \cdots, h_n)^T$. For $r \in \mathbb{R}^+$, $h^T L_n = \mathbf{0}_n^T$ and $\sum_{i=1}^n h_i = 1$. Define $H = \mathrm{diag}(h_1, h_2, \cdots, h_n)$. If the directed graph $\mathcal{G}$ is strongly connected, then the matrix $\mathbf{L} = (HL_n + L_n^T H)/2$ is positive semidefinite. Furthermore, it has only one zero eigenvalue.

B. Proximal Operator

For $x \in \mathbb{R}^r$, if $f(x)$ is a lower semi-continuous convex function, then $prox_f[y]$ of $f(x)$ associated with a point $y \in \mathbb{R}^r$, which is called the proximal operator, can be denoted as

$$\mathrm{prox}_f[y] = \arg\min_x \{f(x) + \frac{1}{2}\|x - y\|^2\}. \tag{1}$$

The subdifferential of $f(x)$ can be denoted as $\partial f(x)$. If $f(x)$ is convex, then $\partial f(x)$ is monotone. For all $x \in \mathbb{R}^r, y \in \mathbb{R}^r$, $z_x \in \partial f(x)$, and $z_y \in \partial f(y)$, there exists that $(z_x - z_y)^T (x - y) \geq 0$. Moreover, the equation $y = \mathrm{prox}_f[x]$ means that

$$y - x \in \partial f(x). \tag{2}$$

III. Problem Statement

The nonsmooth multi-cluster game problem with inter-cluster weight-unbalanced directed graph is formulated in this section. Consider a group of n players $\mathcal{N} = \{1, \cdots, n\}$ involved in this game, which is divided into m clusters. In each cluster $j \in \{1, \cdots, m\}$, n_j players are contained to achieve a consensus decision under a coupled nonsmooth inequality constraint, and $n = \sum_{j=1}^m n_j$. For the player $i \in \{1, \cdots, n_j\}$ in the cluster $j \in \{1, \cdots, m\}$, which is denoted as player $p_{i,j}$, the decision profile is denoted as $x_j^i \in \mathbb{R}^q$. The decision profiles of cluster j are stacked as $x_j = col(x_j^1, \cdots, x_j^{n_j}) \in \mathbb{R}^{n_j q}$. The leaders of the clusters will form an inter-cluster network. In consideration of privacy and communication costs, only the leader of each cluster knows the corresponding local information of resource allocation constraint. Without loss of generality, we designate the leader as player $p_{1,j}$ in each cluster j. These m leaders discuss and decides their local GNE seeking strategies in a inter-cluster network, which is modeled as a weight-unbalanced directed graph $\mathcal{G}_0$. The inner-cluster graph for the cluster $j \in \{1, \cdots, m\}$ is denoted as a undirected graph $\mathcal{G}_j$. Decision profiles of all players except x_j are denoted as $x_{-j} = [(x_1)^T, \cdots, (x_{j-1})^T, (x_{j+1})^T, \cdots, (x_m)^T] \in \mathbb{R}^{(n-n_j)q}$. However, players in cluster $j \in \{1, \cdots, m\}$ can only receive informations of x_{-j} from neighbour clusters $\mathcal{N}_j^{\mathcal{G}_0}$ in $\mathcal{G}_0$. We define $S_j^i(x_{-j})$ as the portion of decision profiles x_{-j} received by player $p_{i,j}$ and $S_j(x_{-j}) = \bigcup_{i=1}^{n_j} S_j^i(x_{-j})$. For the cluster $j \in \{1, \cdots, m\}$, the payoff function $F_j(x_j, x_{-j})$ is defined as $F_j(x_j, x_{-j}) = F_j^1(x_j, x_{-j}) + F_j^2(x_j, x_{-j}) = \sum_{i=1}^{n_j} f_{j,i}(x_j^i, S_j^i(x_{-j}))$, where $f_{j,i}(x_j^i, S_j^i(x_{-j})) = f_{j,i}^1(x_j^i, S_j^i(x_{-j})) + f_{j,i}^2(x_j^i, S_j^i(x_{-j}))$ and $F_j^k(x_j, x_{-j}) = \sum_{i=1}^{n_j} f_{j,i}^k(x_j^i, S_j^i(x_{-j}))$ for $k = 1, 2$. There are two convex functions $f_{j,i}^1, f_{j,i}^2 : \mathbb{R}^q \times \mathbb{R}^k \to \mathbb{R}$ for each player i contained in its local payoff function, where $f_{j,i}^1$ is smooth w.r.t x_j^i, $f_{j,i}^2$ is nonsmooth w.r.t x_j^i. Additionally, $k \in \mathbb{R}$ represents the dimension of $S_j^i(x_{-j})$. For all decision profiles and $j \in \{1, \cdots, m\}$, the feasibility set Ω is defined as

$$\begin{aligned}\Omega = \{x \in \mathbb{R}^{nq} | \sum_{j=1}^m x_j^1 = \sum_{j=1}^m r_j, G_j(x_j) \leq 0, \\ L_{j,q} x_j = \mathbf{0}_{n_j q}\},\end{aligned} \tag{3}$$

where $r_j \in \mathbb{R}^q$, $G_j(x_j) = \sum_{i \in \mathcal{N}_j} g_j^i(x_j^i)$, L_j is the Laplacian matrix of the graph $\mathcal{G}_j$, and $L_{j,q} = L_j \otimes I_q \in \mathbb{R}^{n_j q \times n_j q}$. The local inequality constraint $g_j^i(x_j^i)$ is nonsmooth and available to the player $p_{i,j}$. The feasibility set Ω is a convex and closed set. For a given x_{-j}, the goal of players in the cluster j involved in this multi-cluster game is to minimize the optimization problem

$$\begin{aligned} &min \quad F_j(x_j, x_{-j}) \\ &s.t. \quad (x_j, x_{-j}) \in \Omega. \end{aligned} \tag{4}$$

Remark 1: In problem (4), each cluster eventually sends out a consensus decision through its leader. All clusters then engage a noncooperative game to determine the final decision within an inter-cluster graph. Moreover, in contrast to previous works, this multi-cluster game problem simultaneously considers nonsmooth composite payoff functions, inner-cluster nonsmooth coupled inequality, and inter-cluster directed graph.

According to [31], the definition of the GNE of multi-cluster noncooperative game (4) is presented as follows.

Definition 1: Given a decision profile x^*. If

$$x_j^* \in \underset{(x_j^*, x_{-j}^*) \in \Omega}{argmin} \; F_j(x_j^*, x_{-j}^*) \tag{5}$$

for players in all clusters $j \in \{1, \cdots, m\}$, then x^* is a GNE of the multi-cluster game (4).

According to [31] and [32], the following lemma is proposed to show that the GNE seeking of problem (4) is equivalent to solving a generalized variational inequality (GVI) problem.

Lemma 1: Define $D(x) = col(\partial_{x_1} F_1(x_1, x_{-1}), \cdots, \partial_{x_m} F_m(x_m, x_{-m}))$. Then a GNE of the multi-cluster game (4) for players in all clusters $j \in \{1, \cdots, m\}$ can be obtained by solving the GVI:

$$Find \quad x^* \in \Omega, \langle d^*, x - x^* \rangle \geq 0, \forall x \in \Omega, \tag{6}$$

which means that $-d^* \in N_\Omega(x^*)$, $d^* \in D(x^*)$. $N_\Omega(x^*) = \{x \in \Omega | x^T x^* \leq 0\}$ is the normal cone of Ω at the point x^*.

Remark 2: For each cluster $j \in \{1, \cdots, m\}$, let the feasible direction of the point x_j in Ω be defined as $P_j(x_j) = \{s_j \in \mathbb{R}^n | s_j = y_j - x_j, y_j \in \Omega\}$. The decent direction of the payoff function $F_j(x_j, x_{-j})$ at the point x_j in Ω is defined as $K_j(x_j) = \{s_j \in \mathbb{R}^n | s_j^T d_j < 0, d_j \in D_j(x_j)\}$, where $d = [d_1, \cdots, d_m]$. If x^* is a solution of GVI (6), then $x^* \in \Omega$ and $P_j(x_j^*) \cap K_j(x_j^*) = \varnothing$, i,e, every feasible direction of x_j^* is not a decent direction of $F_j(x_j, x_{-j})$ at the point x_j^* in Ω, which is the definition of the Nash equilibrium of the multi-cluster game problem investigated in our paper.

Assumptions for the well-posedness of the problem (4) are proposed as follows.

Assumption 1:

(1) The payoff function $f^1_{j,i}$ is strongly convex and twice continuously differentiable w.r.t. x^i_j for a given x_{-j}, which means that there exists a constant $K_1 > 0$ for player i such that, $(\nabla_{x^i_j} f^1_{j,i}(y, S^i_j(x_{-j})) - \nabla_{x^i_j} f^1_{j,i}(z, S^i_j(x_{-j})))^T(y-z) \geq K_1\|v_1 - v_2\|^2$, where $y \in \mathbb{R}^q$, $z \in \mathbb{R}^q$, and $y \neq z$ for $i \in \{1, \cdots, n_j\}$, $j \in \{1, \cdots, m\}$. $D(x)$ is monotone.

(2) Proper closed functions $f^2_{j,i}$ and g^i_j are lower semi-continuous and convex. Their proximal operators can be computed easily for $i \in \{1, \cdots, n_j\}$, $j \in \{1, \cdots, m\}$. Furthermore, g^i_j is Lipschitz continuous with $K^i_{2,j}$, $K_2 = \max_{i\in\{1,\cdots,n_j\}, j\in\{1,\cdots,m\}}[K^i_{2,j}]$, and $K_2 > 0$.

(3) The inter-cluster graph $\mathcal{G}_0$ is directed, strongly connected, and weight-unbalanced. The inner-cluster graph $\mathcal{G}_j$ is undirected and connected for all $j \in \{1, \cdots, m\}$.

(4) There exists at least one feasible point to satisfy the Slater's condition of problem (4).

Moreover, the following lemma presents the KKT condition for the solution of GVI (6), which is also a GNE of problem (4) according to Lemma 1.

Lemma 2: If Assumption 1 holds, then $x^* \in \mathbb{R}^{nq}$ can be a GNE of the problem (4) if $\alpha_1 \in \mathbb{R}^+$, $\alpha_2 \in \mathbb{R}$, $\alpha_3 \in \mathbb{R}$, $\mu_j \in \mathbb{R}^{n_j q}$, $v^0 \in \mathbb{R}^q$, $v^*_j \in \mathbb{R}^{n_j q}$, and $\omega^* \in \mathbb{R}^{mq}$ exist to satisfy that

$$\begin{cases} \mathbf{0}_{n_j q} \in \alpha_1 \nabla_{x_j} F^1_j(x^*_j, x^*_{-j}) + \alpha_1 \partial_{x_j} F^2_j(x^*_j, x^*_{-j}) \\ \qquad + \alpha_2 \partial_{x_j}(\omega^*_j g_j(x^*_j)) + \alpha_3 L_j \mu^*_j - v^*_j, \\ \mathbf{L}_j x^*_j = \mathbf{0}_{n_j q}, G_j(x^*_j) \leq 0, (\omega^*)^T g(x^*) = 0, \\ L_j \omega^*_j = \mathbf{0}_{n_j}, \sum_{j=1}^m x^{1*}_j = \sum_{j=1}^m r_j, v^{1*}_j = v_0, \end{cases} \tag{7}$$

where $j \in \{1, \cdots, m\}$, $x^{1*} = [x^{1*}_1, \cdots, x^{1*}_m]^T \in \mathbb{R}^{mq}$, $g(x^*_j) = [g^1_j(x^{1*}_j), \cdots, g^{n_j}_j(x^{n_j*}_j)]^T \in \mathbb{R}^{n_j}$, $g(x^*) = [g_1(x^*_1), g_2(x^*_2), \cdots, g_m(x^*_m)]^T \in \mathbb{R}^n$, and $v^{i*}_j = 0$ for $i \neq 1$.

The proof of Lemma 2 can be deduce straightforwardly.

IV. GNE Seeking Algorithm Design

A distributed Lipschitz-continuous GNE seeking algorithm is proposed in this section for the directed nonsmooth multi-cluster game (4). The proposed algorithm is designed with the following mechanism:

$$\begin{aligned} \dot{x}^i_j =& \text{prox}_{\alpha_1 f^2_{i,j}}\big[x^i_j - \alpha_1 \nabla_{x^i_j} f^1_{i,j}(x^i_j, S^i_j(x_{-j})) \\ & - \alpha_3 \sum_{k\in\mathcal{N}^{\mathcal{G}_j}_i} a^j_{i,k}(\mu^i_j - \mu^i_j) - \alpha_3 \sum_{k\in\mathcal{N}^{\mathcal{G}_j}_i} a^j_{i,k}(x^i_j - x^k_j) \\ & + \alpha_4 y^i_j + \bar{v}^i_j\big] - x^i_j, \\ \dot{y}^i_j =& \text{prox}_{\alpha_2 \omega^i_j g^i_j}\big[x^i_j - \alpha_4 y^i_j\big] - x^i_j, \\ \dot{\omega}^i_j =& \mathbf{P}_{\mathbb{R}_+}[\omega^i_j + \alpha_2 g^i_j(x^i_j + \dot{y}^i_j) - \alpha_6 \sum_{k\in\mathcal{N}^{\mathcal{G}_j}_i} a^j_{i,k}(\omega^i_j - \omega^k_j) \\ & - \alpha_6 \sum_{k\in\mathcal{N}^{\mathcal{G}_j}_i} a^j_{i,k}(\xi^i_j - \xi^k_j)] - \omega^i_j, \\ \dot{\mu}^i_j =& \alpha_3 \sum_{k\in\mathcal{N}^{\mathcal{G}_j}_i} a^j_{i,k}(x^i_j - x^k_j), \dot{\xi}^i_j = \alpha_6 \sum_{k\in\mathcal{N}^{\mathcal{G}_j}_i} a^j_{i,k}(\omega^i_j - \omega^k_j), \\ \dot{v}_j =& -\frac{1}{h_j}(x^1_j + \dot{x}^1_j - r_j) - \alpha_5 \sum_{l\in\mathcal{N}^{\mathcal{G}_0}_j} a^0_{j,l}(v_j - v_l) - \eta_j, \\ \dot{\eta}_j =& \alpha_5 \sum_{l\in\mathcal{N}^{\mathcal{G}_0}_j} a^0_{j,l}(v_j - v_l), \quad \eta_j(0) = \mathbf{0}_q, \\ \dot{\omega}_j =& P_{\bar{\mathbb{R}}_+}[\omega_j + \alpha_2 g_j(x^1_j + \dot{y}_j)] - \omega_j, \end{aligned} \tag{8}$$

where $\bar{v}^1_j = v_j$ and $\bar{v}^i_j = \mathbf{0}_q$ for $i \neq 1$.

A compact form of the algorithm (8) is presented as follows.

$$\begin{aligned} \dot{x} =& \text{Prox}_{\alpha_1 F^2}\big[x - \alpha_1 \nabla F^1(x) - \alpha_3 \mathbf{L}_{n,q}\mu \\ & - \alpha_3 \mathbf{L}_{n,q} x + \alpha_4 y + \bar{v}\big] - x, \\ \dot{y} =& \text{Prox}_{\alpha_2 \omega^T g}[x - \alpha_4 y] - x, \\ \dot{\omega} =& \mathbf{P}_{\mathbb{R}^n_+}[\omega + \alpha_2 g(x + \dot{y}) - \alpha_6 \mathbf{L}\omega - \alpha_6 \mathbf{L}\xi] - \omega, \\ \dot{\mu} =& \alpha_3 \mathbf{L}_{n,q} x, \dot{\xi} = \alpha_6 \mathbf{L}\omega, \\ \dot{v} =& -H^{-1}(x^1 + \dot{x}^1 - r) - \alpha_5 L_0 v - \eta, \\ \dot{\eta} =& \alpha_5 L_0 v, \eta(0) = \mathbf{0}_{mq}, \end{aligned} \tag{9}$$

where $\nabla_{x_j} F^k_j(x_j, x_{-j}) = col(\nabla_{x^1_j} f^k_{1,j}(x^1_j, S^1_j(x_{-j})), \cdots, \nabla_{x^{n_j}_j} f^k_{n_j,j}(x^{n_j}_j, S^{n_j}_j(x_{-j})))$, $\nabla F^1(x) = col(\nabla_{x_1} F^1_1(x_1, x_{-1}), \cdots, \nabla_{x_m} F^1_m(x_m, x_{-m}))$, $\partial F^2(x) = col(\partial_{x_1} F^2_1(x_1, x_{-1}), \cdots, \partial_{x_m} F^2_m(x_m, x_{-m}))$, $x = col(x_1, \cdots, x_m) \in \mathbb{R}^{nq}$, $x^1 = col(x^1_1, \cdots, x^1_m) \in \mathbb{R}^{mq}$, $\mu = col(\mu_1, \cdots, \mu_m) \in \mathbb{R}^{nq}$, $\mu = col(\mu_1, \cdots, \mu_m) \in \mathbb{R}^{nq}$, $v = col(v_1, \cdots, v_m) \in \mathbb{R}^{mq}$, $\bar{v} = col(\bar{v}_1, \cdots, \bar{v}_m) \in \mathbb{R}^{nq}$, $y_j = col(y^1_j, \cdots, y^{n_j}_j) \in \mathbb{R}^{n_j q}$, $y = col(y_1, \cdots, y_m) \in \mathbb{R}^{nq}$, $\omega_j = col(\omega^1_j, \cdots, \omega^{n_j}_j) \in \mathbb{R}^{n_j}$, $\omega = col(\omega_1, \cdots, \omega_m) \in \mathbb{R}^n$, $\xi_j = col(\xi^1_j, \cdots, \xi^{n_j}_j) \in \mathbb{R}^{n_j}$, $\xi = col(\xi_1, \cdots, \xi_m) \in \mathbb{R}^n$, $g(x) = col(g_1(x_1), \cdots, g_m(x_m)) \in \mathbb{R}^n$, $r = col(r_1, \cdots, r_m) \in \mathbb{R}^{mq}$, $\eta = col(\eta_1, \cdots, \eta_m) \in \mathbb{R}^{mq}$, $H^{-1} = diag(1/h_1, \cdots, 1/h_m) \in \mathbb{R}^{m\times m}$, $r = col(r_1, \cdots, r_n) \in \mathbb{R}^{mq}$, $\mathbf{L} = diag(L_1, \cdots, L_m)$, and $\mathbf{L}_{n,q} = \mathbf{L} \otimes I_q$.

Remark 3: The motivation of the design of algorithm (9) is to seek the saddle-point of the modified Lagrangian function as

$$L(x, \omega, \mu, v) = F(x) + \omega^T g(x) + \mu^T \mathbf{L}_{n,q} x - v^T(x - r). \tag{10}$$

The auxiliary variable y and the parameter α_4 are presented in algorithm (9) to separate $\mathbf{0}_{n_j q} \in \alpha_1 \nabla_{x_j} F^1_j(x^*_j, x^*_{-j}) +$

$\alpha_1\partial_{x_j}F_j^2(x_j^*, x_{-j}^*)$ $+\alpha_2\partial_{x_j}(\omega_j^* g_j(x_j^*)) + \alpha_3 L_j\mu_j^* - v_j^*$ in (7) to two part as

$$\begin{aligned}&\alpha_1\nabla_{x_j}F_j^1(x_j^*, x_{-j}^*) + \alpha_4 y_j^* + \alpha_3 L_j\mu_j^*\\ &- v_j^* \in -\alpha_1\partial_{x_j}F_j^2(x_j^*, x_{-j}^*),\end{aligned} \tag{11}$$

and

$$\alpha_4 y_j^* \in -\alpha_2\partial_{x_j}(\omega_j^* g_j(x_j^*)), \tag{12}$$

where $Prox_{\alpha_1 F^2}[\cdot]$ and $Prox_{\alpha_2\omega^T g}[\cdot]$ w.r.t. x are designed to fulfil the requirements of (11) and (12). For minimizing the Lagrangian function in the x-direction, the first equation in algorithm (9) is derived with the proximal operator of $\alpha_1 F^2(x)$, and the second equation in algorithm (9) is derived with the proximal operator of $\alpha_2\omega^T g$. For maximizing the Lagrangian function in the ω-direction, μ-direction, and v-direction, the third equation, fourth equation, and fifth equation in algorithm (9) are derived respectively.

V. Convergence Result

The convergence result of algorithm (9) is deduced in this section. First, the relationship between the equilibrium of algorithm (9) and the solution of problem (4) can be presented as the following Lemma 3.

Lemma 3: Under Assumption 1, if (x^*, y^*, ω^*, μ^*, ξ^*, v^*, η^*) is an equilibrium of algorithm (9), then x^* can be an GNE for the game (4).

Proof: Suppose (x^*, y^*, ω^*, μ^*, ξ^*, v^*, η^*) is an equilibrium of algorithm (9). Combining (2) and algorithm (9), it shows that:

$$x^* - \alpha_1\nabla F^1(x^*) - \alpha_3\mathbf{L}\mu^* + \alpha_4 y^* + \bar{v}^* \in \alpha_1\partial F^2(x^*), \tag{13a}$$

$$-\alpha_4 y^* \in \alpha_2\partial_x((\omega^*)^T g(x^*)), \tag{13b}$$

$$\mathbf{P}_{\bar{\mathbb{R}}_+^n}[\omega^* + \alpha_2 g(x^*) - \alpha_6\mathbf{L}\xi^*] = \omega^*, \omega^* \geq \mathbf{0}_n \tag{13c}$$

$$-H^{-1}(x^* - r) - \alpha_5 L_0 v^* - \eta^* = \mathbf{0}_{nq}, \tag{13d}$$

$$L_0 v^* = \mathbf{0}_{mq}, \mathbf{L}_{n,q}x^* = \mathbf{0}_{nq}, \mathbf{L}\omega^* = \mathbf{0}_n, \tag{13e}$$

which means that $\mathbf{0}_{nq} \in x^* - \alpha_1\nabla F^1(x^*) - \alpha_1\partial F^2(x^*) - \alpha_3\mathbf{L}\mu^* - \alpha_3\mathbf{L}x^* + \alpha_4\bar{y}^* + \bar{v}^*$.

Considering (13c), if $\omega^* = \mathbf{0}_n$, there exists that $\alpha_2 g(x^*) - \alpha_6\mathbf{L}\xi^* \leq \mathbf{0}_m$, which means that $G(x^*) < 0$; if $\omega^* > \mathbf{0}_m$, it follows that $\alpha_2 g(x^*) - \alpha_6\mathbf{L}\xi^* = \mathbf{0}_m$, which means that $G(x^*) = 0$. Hence, from the above, $G(x^*) \leq 0$ and $(\omega^*)^T[g(x^*)] = 0$. From (13d) and (13e), it yields that $-H^{-1}(x^* - r) - \alpha_5 L_0 v^* - \eta^* = \mathbf{0}_{nq}$. Remember that $\eta_j(0) = \mathbf{0}_q$, which means that $\sum_{j=1}^m h_j\dot{\eta}_j = \alpha_5\mathbf{1}^T H L_0 v = \mathbf{0}_q$ and $\sum_{j=1}^m h_j\eta_j = \sum_{j=1}^m h_j\eta_j(0) = \mathbf{0}_q$. Then, it holds that $\sum_{i=1}^n(x_j^{1*} - r_j) = -\sum_{j=1}^m h_j\eta_j^* = \mathbf{0}_q$. According to Lemma 2, x^* is a GNE of problem (4). ■

Then the following Lemma regarding the relation between the Lagrangian multiplier ω, the nonsmooth function $g(x)$, and the decision profile x is presented. This lemma is derive according to the game (4) and algorithm (9), which is an extension of Theorem 3.1 in [23] for the problem (4) using algorithm (9).

Lemma 4: Under Assumption 1, if (x^*, y^*, ω^*, μ^*, ξ^*, v^*, η^*) is an equilibrium of (9), then it holds that

$$\begin{aligned}&[\partial_x(\omega^T g(x+\dot{y})) - \partial_x((\omega^*)^T g(x^*))]^T(\hat{x}^* + \dot{y})\\ &\qquad\qquad -(g(x+\dot{y}) - g(x^*))^T\hat{\omega}^* \geq 0,\end{aligned} \tag{14}$$

where $\hat{\omega}^* = \omega - \omega^*$ and $\hat{x}^* = x - x^*$.

The Lyapunov candidate is designed as $V(x,\ y,\ \omega,\ \mu,\ \xi,\ v,\ \eta) = V_1(x,y) + V_2(x,\mu,\omega,\xi) + V_3(v,\eta)$, and it can be expressed as

$$\begin{aligned}V_1(x,y) =& \frac{\zeta{+}1}{2}\big[\|\hat{x}^*\|^2 + \alpha_4(\|\hat{y}^*\|^2 - 2(\hat{x}^*)^T\hat{y}^*)\\ &+2\alpha_1[F^1(x) - F^1(x^*) - (\hat{x}^*)^T\nabla F^1(x^*)]\big],\\ V_2(x,\mu,\omega,\xi) =& \frac{\zeta{+}1}{2}\big[\|\hat{\mu}^*\|^2 + \alpha_3 x^T\mathbf{L}x + 2\alpha_3 x\mathbf{L}\hat{\mu}^*\\ &+\|\hat{\omega}^*\|^2 + \|\hat{\xi}^*\|^2 + \alpha_6\omega^T\mathbf{L}\omega + 2\alpha_6\omega\mathbf{L}\hat{\xi}\big],\\ V_3(v,\eta) =& \frac{1}{2}[\zeta(\hat{v}^*)^T H\hat{v}^* + (\hat{v}^* + \hat{\eta}^*)^T H(\hat{v}^* + \hat{\eta}^*)],\end{aligned} \tag{15}$$

where $\alpha_6 > 0$, $\hat{y}^* = y - y^*$, $\hat{\mu}^* = \mu - \mu^*$, $\hat{\xi}^* = \xi - \xi^*$, $\hat{v}^* = v - v^*$, and $\hat{\eta}^* = \eta - \eta^*$.

Theorem 1: Consider Assumption (1) and algorithm (9). If $\zeta > \frac{1}{h^*} - 1$, $\alpha_1 > \frac{2}{K_1}$, $0 < \alpha_2 < \min[\frac{1}{(K_2)^2}, 2]$, $0 < \alpha_3 < 1/(n_{max} + 1)$, $0 < \alpha_4 < 1 - \max[\alpha_2(K_2)^2, \alpha_3 n_{max}]$, $\alpha_5 > \frac{(\zeta+1)^2}{\zeta\lambda_{min}(\hat{L}_0)}$, and $0 < \alpha_6 < \frac{1}{n_{max}}$, where $n_{max} = \max_{j\in\{1,\cdots,m\}}[m_j]$, then

(1) $V(x,y,\omega,\mu,\xi,v,\eta) \geq 0$. $V(x,y,\omega,\mu,\xi,v,\eta) = 0$ iff $(x,y,\omega,\mu,\xi,v,\eta) = (x^*,y^*,\omega^*,\mu^*,\xi^*,v^*,\eta^*)$.

(2) The trajectory $(x,y,\omega,\mu,\xi,v,\eta)$ is bounded. Furthermore, x converges to the GNE of game (4) as $t \to \infty$.

Proof: (1) It is trivial to obtain that $V(x^*,y^*,\omega^*,\mu^*,\xi^*,v^*,\eta^*) = 0$. The following statement is about to proof that $V(x,y,\omega,\mu,\xi,v,\eta) > 0$ when $(x,y,\omega,\mu,\xi,v,\eta) \neq (x^*,y^*,\omega^*,\mu^*,\xi^*,v^*,\eta^*)$. Combining α_6 and (15), we have

$$\begin{aligned}&V_2(x,\mu,\omega,\xi)\\ \geq& \frac{\zeta{+}1}{2}\big[\|\hat{\mu}^*\|^2 + 2\alpha_3 x\mathbf{L}\hat{\mu}^* + \|\hat{\omega}^*\|^2 + \|\hat{\xi}^*\|^2 + 2\alpha_6\omega\mathbf{L}\hat{\xi}\big]\\ \geq& \frac{\zeta{+}1}{2}\big[(1 - \alpha_3\lambda_{max}(\mathbf{L}))\|\hat{\mu}^*\|^2 - \alpha_3\lambda_{max}(\mathbf{L}))\|\hat{x}^*\|^2\big].\end{aligned} \tag{16}$$

Since $F^1(x)$ is a strongly convex function, there exists that $F^1(x) - F^1(x^*) - (\hat{x}^*)^T\nabla F^1(x^*) \geq 0$. It can be also easily to show that $V_3(v,\eta) \geq 0$. Recalling $0 < \alpha_3 < \frac{1}{n_{max}+1}$, $\lambda_{max}(\mathbf{L}_{n,q}) \leq n_{max}$, $0 < \alpha_4 < 1 - \max[\alpha_2(K_2)^2, \alpha_3 n_{max}]$, and (16) with (15), it yields that

$$\begin{aligned}&V(x,y,\omega,\mu,\xi,v,\eta)\\ \geq& \frac{\zeta{+}1}{2}\big[\|\hat{x}^*\|^2 + \alpha_4(\|\hat{y}^*\|^2 - 2(\hat{x}^*)^T\hat{y}^*)\\ &- \alpha_3\lambda_{max}(\mathbf{L})\|\hat{x}^*\|^2\big]\\ \geq& \frac{\zeta{+}1}{2}\big[\kappa_1\|\hat{x}^*\|^2 + \alpha_4(\|\hat{y}^*\|^2 - 2(\hat{x}^*)^T\hat{y}^*)\big]\\ \geq& \frac{\zeta{+}1}{2}\big[\kappa_1\big[\|\hat{x}^* - \kappa_2\hat{y}^*\|^2 + \kappa_2(1 - \kappa_2)\|\hat{y}^*\|^2\big]\\ \geq& 0,\end{aligned} \tag{17}$$

where $\kappa_1 = 1 - \alpha_3\lambda_{max}(\mathbf{L}) \geq 1 - \frac{n_{max}}{n_{max}+1} > 0$ and $\kappa_2 = \alpha_4/\kappa_1 > 0$. Therefore, $V(x,y,\omega,\mu,\xi,v,\eta)$ is positive definite, radially unbounded, $V(x,y,\omega,\mu,\xi,v,\eta) \geq 0$ and is zero iff $(x,y,\omega,\mu,\xi,v,\eta) = (x^*,y^*,\omega^*,\mu^*,\xi^*,v^*,\eta^*)$.

(2) Here it is proved that $\dot{V}(t) \leq 0$ for any $t \geq 0$. From (9), it is shown that

$$\begin{aligned} x + \dot{x} =& \mathrm{Prox}_{\alpha_1 F^2}[x - \alpha_1 \nabla F^1(x) - \alpha_3 \mathbf{L}_{n,q}\mu \\ & - \alpha_3 \mathbf{L}_{n,q} x + \alpha_4 y + \bar{v}], \end{aligned} \tag{18a}$$

$$\begin{aligned} x^* =& \mathrm{Prox}_{\alpha_1 F^2}[x^* - \alpha_1 \nabla F^1(x^*) - \alpha_3 \mathbf{L}_{n,q}\mu^* \\ & + \alpha_4 y^* + \bar{v}^*], \end{aligned} \tag{18b}$$

$$x + \dot{y} = \mathrm{Prox}_{\alpha_2 \omega^T g}[x - \alpha_4 y], \tag{18c}$$

$$x^* = \mathrm{Prox}_{\alpha_4 (\omega^*)^T g}[x^* - \alpha_4 y^*], \tag{18d}$$

$$\omega + \dot{\omega} = \mathrm{P}_{\mathbb{R}^n_+}[\omega + \alpha_2 g(x+\dot{y}) - \alpha_6 \mathbf{L}\omega - \alpha_6 \mathbf{L}\xi], \tag{18e}$$

$$\omega^* = \mathrm{P}_{\mathbb{R}^n_+}[\omega^* + \alpha_2 g(x^*) - \alpha_6 \mathbf{L}\xi^*]. \tag{18f}$$

According to (18a) and (18b), we can have that

$$\begin{aligned} & - \alpha_1 \nabla F^1(x) - \alpha_3 \mathbf{L}_{n,q}\mu - \alpha_3 \mathbf{L}_{n,q} x \\ & + \alpha_4 y + \bar{v} - \dot{x} \in \alpha_1 \partial F^2(x + \dot{x}), \end{aligned} \tag{19}$$

with

$$\begin{aligned} & - \alpha_1 \nabla F^1(x^*) - \alpha_3 \mathbf{L}_{n,q}\mu^* \\ & + \alpha_4 \bar{y}^* + \bar{v}^* \in \alpha_1 \partial F^2(x^*). \end{aligned} \tag{20}$$

Similarly, from (18c) and (18d), it yields that

$$\begin{aligned} & -\alpha_4 y - \dot{y} \in \alpha_2 \partial_x(\omega^T g(x)), \\ & -\alpha_4 y^* \in \alpha_2 \partial_x((\omega^*)^T g(x^*)). \end{aligned} \tag{21}$$

Moreover, (18e) and (18f) imply that

$$(\hat{\omega}+\dot{\omega})^T\big[\alpha_2 g(x+\dot{y}) - \alpha_2 g(x^*) - \alpha_6 \mathbf{L}(\hat{\omega}^* + \hat{\xi}^*) - \dot{\omega}\big] \geq 0. \tag{22}$$

Since the subdifferential $\partial F^2(x)$ is monotone caused by the convexity of $F^2(x)$, it can be shown from (2), (19), and (20) that

$$\begin{aligned} & \big[- \alpha_1 \nabla F^1(\hat{x}^*) - \alpha_3 \mathbf{L}_{n,q}\hat{x}^* - \alpha_3 \mathbf{L}_{n,q}\hat{\mu}^* \\ & + \alpha_4 \hat{y}^* + \hat{\bar{v}}^* - \dot{x}\big]^T (\hat{x}^* + \dot{x}) \geq 0, \end{aligned} \tag{23}$$

where $\nabla F^1(\hat{x}^*) = \nabla F^1(x) - \nabla F^1(x^*)$ and $\hat{\bar{v}}^* = \bar{v} - \bar{v}^*$.

From (23), it can be derived that

$$\begin{aligned} & (\hat{x}^*)^T \dot{x} + \alpha_1 \dot{x}^T \nabla F^1(\hat{x}^*) + \alpha_3 \dot{x}^T \mathbf{L}_{n,q}\hat{x}^* \\ & + \alpha_3 (\hat{\mu}^*)^T \mathbf{L}_{n,q}\hat{x}^* + \alpha_3 \dot{x}^T \mathbf{L}_{n,q}\hat{\mu}^* - \alpha_4 (\hat{y}^*)^T \dot{x} \\ \leq & - \|\dot{x}\|^2 - \alpha_3 (\hat{x}^*)^T \mathbf{L}_{n,q}\hat{x} - \alpha_1 (\hat{x}^*)^T \nabla F^1(\hat{x}^*) \\ & + (\hat{x}^* + \dot{x})^T \hat{\bar{v}}^* + \alpha_4 (\hat{y}^*)^T \hat{x}^*. \end{aligned} \tag{24}$$

Combining Lemma 4 and (21), it yields that

$$(-\alpha_4 \hat{y}^* - \dot{y})^T (\hat{x}^* + \dot{y}) - \alpha_2 (\hat{\omega}^*)^T (g(x+\dot{y}) - g(x^*)) \geq 0, \tag{25}$$

which means that

$$\begin{aligned} \alpha_4 (\hat{y}^*)^T \dot{y} \leq & - (\hat{x}^*)^T \dot{y} - \alpha_1 (\hat{\omega}^*)^T (g(x+\dot{y}) - g(x^*)) \\ & - \|\dot{y}\|^2 - \alpha_4 (\hat{y}^*)^T \hat{x}^*. \end{aligned} \tag{26}$$

According to (22), it follows that

$$\begin{aligned} & (\hat{\omega}^*)^T \dot{\omega} + \alpha_6 (\hat{\xi}^*)^T \mathbf{L}\hat{\omega}^* + \alpha_6 \dot{\omega}^T \mathbf{L}\hat{\omega}^* + \alpha_6 \dot{\omega}^T \mathbf{L}\hat{\xi}^* \\ \leq & (\hat{\omega}^* + \dot{\omega})^T [\alpha_2 (g(x+\dot{y}) - g(x^*))] - \|\dot{\omega}\|^2 - \alpha_6 (\hat{\omega}^*)^T \mathbf{L}\hat{\omega}^*. \end{aligned} \tag{27}$$

Considering (24), (26), and (27) together with (15), it can be presented that

$$\begin{aligned} & \dot{V}(x,y,\omega,\mu,\xi,v,\eta) \\ =& (\zeta+1)\big[(\hat{x}^*)^T \dot{x} + \alpha_4 (\hat{y}^*)^T \dot{y} - \alpha_4 (\hat{x}^*)^T \dot{y} - \alpha_4 (\hat{y}^*)^T \dot{x} \\ & + \alpha_1 \dot{x}^T \nabla F^1(\hat{x}^*)\big] + (\zeta+1)\big[\alpha_3 \dot{x}^T \mathbf{L}_{n,q}(\hat{x}^* + \hat{\mu}^*) \\ & + (\hat{\mu}^*)^T \dot{\mu} + (\hat{\omega}^*)^T \dot{\omega} + (\hat{\xi}^*)^T \dot{\xi} + \alpha_6 \dot{\omega}^T \mathbf{L}\hat{\omega}^* + \alpha_6 \dot{\omega} \mathbf{L}\hat{\xi}^*\big] \\ & + (\zeta+1)\big[(\alpha_3)^2 (\hat{x}^*)^T (\mathbf{L}_{n,q})^2 \hat{x}^* + (\alpha_6)^2 (\hat{\omega}^*)^T (\mathbf{L})^2 \hat{\omega}^*\big] \\ & + \big[\zeta (\hat{v}^*)^T H \dot{v} + (\hat{v}^* + \hat{\eta}^*)^T H (\dot{v} + \dot{\eta})\big] \\ =& (\zeta+1)\big[(\hat{x}^*)^T \dot{x} + \alpha_4 (\hat{y}^*)^T \dot{y} - \alpha_4 (\hat{y}^*)^T \dot{x} + \alpha_1 \dot{x}^T \nabla F^1(\hat{x}^*) \\ & + \alpha_3 \dot{x}^T \mathbf{L}_{n,q}\hat{x}^* + \alpha_3 (\hat{\mu}^*)^T \mathbf{L}_{n,q}\hat{x}^* + \alpha_3 \dot{x}^T \mathbf{L}_{n,q}\hat{\mu}^*\big] \\ & + (\zeta+1)\big[(\hat{\omega}^*)^T \dot{\omega} + \alpha_6 (\hat{\xi}^*)^T \mathbf{L}\hat{\omega}^* + \alpha_6 \dot{\omega}^T \mathbf{L}(\hat{\omega}^* + \hat{\xi}^*)\big] \\ & + (\zeta+1)\big[(\alpha_3)^2 (\hat{x}^*)^T (\mathbf{L}_{n,q})^2 \hat{x}^* + (\alpha_6)^2 (\hat{\omega}^*)^T (\mathbf{L})^2 \hat{\omega}^* \\ & - \alpha_4 (\hat{x}^*)^T \dot{y}\big] + \big[\zeta (\hat{v}^*)^T H \dot{v} + (\hat{v}^* + \hat{\eta}^*)^T H (\dot{v} + \dot{\eta})\big] \\ \leq & - (\zeta+1)\big[\|\dot{x}\|^2 + \|\dot{y}\|^2 + \alpha_1 (\hat{x}^*)^T \nabla F^1(\hat{x}^*) + \|\dot{\omega}\|^2\big] \\ & - \alpha_5 \zeta (\hat{v}^*)^T \hat{L}_0 \hat{v}^* - (\hat{\eta}^*)^T H \hat{\eta}^* - (\dot{x}^1)^T \eta^* - (\hat{x}^{1*})^T \hat{\eta}^* \\ & - (\zeta+1)\big[(\hat{v}^*)^T H \hat{\eta}^* + (\alpha_4 + 1)(\hat{x}^*)^T \dot{y} \\ & - \alpha_2 \dot{\omega}^T (g(x^1 + \dot{y}) - g(x^{1*}))\big], \end{aligned} \tag{28}$$

where $\hat{L}_0 = \frac{1}{2}(H\mathbf{L}_0 + \mathbf{L}_0 H)$, and $(\hat{x}^*)^T \nabla F^1(\hat{x}^*) \geq K_1 \|\hat{x}^*\|^2$ according to Assumption 1.

Define $v = v^a + v^b$, where v^a is the vector of ones, and v^b is not. Then $(\hat{v}^*)^T H \hat{\eta}^* = (v^b)^T H \hat{\eta}^*$, since $(v^a)^T H \hat{\eta}^* = 0$ caused of $\mathbf{1}_m^T H \dot{\eta} = 0$. Note that $(\hat{x}^*)^T \dot{y} \leq \frac{1}{2}\|\hat{x}^*\|^2 + \frac{1}{2}\|\dot{y}\|^2$, $(\zeta+1)(v^b)^T H \hat{\eta}^* \leq (\zeta+1)^2 \|v^b\|^2 + \frac{1}{4}(\hat{\eta}^*)^T H \hat{\eta}^*$, $(\dot{x}^1)^T \hat{\eta}^* \leq \frac{1}{h^*}\|\dot{x}\|^2 + \frac{1}{4}(\hat{\eta}^*)^T H \hat{\eta}^*$, $(\hat{x}^{1*})^T \hat{\eta}^* \leq \frac{1}{h^*}\|\hat{x}^*\|^2 + \frac{1}{4}(\hat{\eta}^*)^T H \hat{\eta}^*$, and $\dot{\omega}^T (g(x+\dot{y}) - g(x)) \leq \frac{1}{2}\|\dot{\omega}\|^2 + \frac{1}{2}K_2^2 \|\dot{y}\|^2$. Hence according to assumptions of α_k for $k \in \{1, \cdots, 5\}$, it is derived from (28) that

$$\begin{aligned} & \dot{V}(x,y,\omega,\mu,\xi,v,\eta) \\ \leq & - (\zeta+1)\big[\|\dot{x}\|^2 + \|\dot{y}\|^2 + \alpha_1 K_1 \|\hat{x}^*\|^2 + \|\dot{\omega}\|^2\big] \\ & - \alpha_5 \zeta (\hat{v}^*)^T \hat{L}_0 \hat{v}^* - \frac{1}{4}(\hat{\eta}^*)^T H \hat{\eta}^* + \frac{1}{h^*}\|\dot{x}\|^2 \\ & + (\zeta+1)^2 \|v^b\|^2 - (\zeta+1)\big[(\alpha_4 + 1)(\hat{x}^*)^T \dot{y} \\ & - \alpha_2 \dot{\omega}^T (g(x^1 + \dot{y}) - g(x^{1*}))\big] + \frac{1}{h^*}\|\hat{x}^*\|^2 \\ \leq & - \epsilon_1 \|\dot{x}\|^2 - \epsilon_2 \|\dot{\omega}\|^2 - \epsilon_3 \|v^b\|^2 - \epsilon_4 \|\dot{y}\|^2 \\ & - \epsilon_5 \|\hat{x}^*\|^2 - \frac{1}{4}(\hat{\eta}^*)^T H \hat{\eta}^* \\ \leq & 0, \end{aligned} \tag{29}$$

where $\varepsilon_1 = \zeta + 1 - \frac{1}{h^*}$, $\varepsilon_2 = (\zeta+1)(1 - \frac{1}{2}\alpha_2)$, $\varepsilon_3 = \alpha_5 \zeta \lambda_{min}(\hat{L}_0) - (\zeta+1)^2$, $\varepsilon_4 = \frac{1}{2}(\zeta+1)(1 - \alpha_4 - \alpha_2 (K_2)^2)$, and $\epsilon_5 = \frac{1}{2}(\zeta+1)(2\alpha_1 K_1 - \alpha_4 - 1) - \frac{1}{h^*}$. Since $V(x,y,\omega,\mu,\xi,v,\eta)$ is positive-definite, radially unbounded, lower bounded, $(x^*,y^*,\omega^*,\mu^*,\xi^*,v^*,\eta^*)$ is Lyapunov stable and the trajectory $(x,y,\omega,\mu,\xi,v,\eta)$ is bounded.

Define a set as $T = \{(x,y,\omega,\mu,\xi,v,\eta) : \dot{V}(x,y,\omega,\mu,\xi,v,\eta) = 0\}$, which implies that $T \subseteq \{(x,y,\omega, \mu, \xi, v, \eta) : \dot{x} = \mathbf{0}_{nq}, \dot{y} = \mathbf{0}_{nq}, \dot{\omega} = \mathbf{0}_n, v^b = \mathbf{0}_{mq}, x = x^*, \eta = \eta^*\}$. Suppose D is the largest invariant set of T. According to the invariance principle, the trajectory $(x,y,\omega,\mu,\xi,v,\eta)$ steered by algorithm (9) converges to

D for $t \to \infty$. If $(x, y, \omega, \mu, \xi, v, \eta)$ is a trajectory of algorithm (9) starting from $(x_0, y_0, \omega_0, \mu_0, \xi_0, v_0, \eta_0) \in D$, $(x, y, \omega, \mu, \xi, v, \eta) \subset D$ for all $t \geq 0$. Hence $\dot{x} = \mathbf{0}_{nq}$ and $\dot{\omega} = \mathbf{0}_{nq}$, which means that $\dot{\mu} \equiv \alpha_3 \mathbf{L}_{n,q} x_0 \equiv P_1 \in \mathbb{R}^{nq}$ and $\dot{\zeta} \equiv \alpha_6 \mathbf{L} \omega_0 \equiv P_2 \in \mathbb{R}^n$. If $P_1 \not\equiv \mathbf{0}_{nq}$ and $P_2 \not\equiv \mathbf{0}_n$, μ and ζ will not stay as constants, which contradicts the invariance principle. Furthermore, with $x \equiv x^*$, $\eta \equiv \eta^*$, and $v^b \equiv \mathbf{0}_{mq}$, it is deduced that $\dot{\eta} \equiv \alpha_5 L_0 v \equiv \mathbf{0}_{mq}$ and $\dot{v} \equiv -(x^{1*} - r) - \eta^* \equiv \mathbf{0}_{mq}$ according to the proof of Lemma 3. Therefore, $D \subseteq \{(x, y, \omega, \mu, \xi, v, \eta) : \dot{x} = \mathbf{0}_{nq}, \dot{\mu} = \mathbf{0}_{nq}, \dot{y} = \mathbf{0}_{nq}, \dot{\omega} = \mathbf{0}_n, \dot{\xi} = \mathbf{0}_n, \dot{v} = \mathbf{0}_{mq}, \dot{\eta} = \mathbf{0}_{mq}\}$. According to Lemma 2 in [27], the limitation of trajectory $(x, y, \omega, \mu, \xi, v, \eta)$ is an equilibrium point to (9). The according to Lemma 3, x converges to the GNE of game (4) as $t \to \infty$.

Remark 4: Parameters α_1 to α_6 in Theorem 1 are provided as sufficient conditions for guaranteeing the convergence of algorithm (9), as deduced of Theorem 1. Parameters h and $\lambda_{min}(\hat{L}_0)$ have been adopted in the design of algorithm (9) with the inter-cluster directed unbalanced graph. Before executing the algorithm (9), some distributed algorithms in [33], [34] can be introduced to estimate these parameters. ⧫

VI. Simulation Results

In this section, we investigate some simulation results to validate the proposed algorithm (9). Consider a distributed multi-agent game with sixteen first-order agents forming four clusters. The local optimization problem of the j-th cluster is defined as follows:

$$\begin{aligned} \min_{x_j \in \mathbb{R}^{2n_j}} \quad & F_j(x_j, x_{-j}), \\ s.t. \quad & G_j(x_j) \leq 0, \quad \mathbf{L}_{j,2} x_j = \mathbf{0}_{2n_j}. \end{aligned} \tag{30}$$

The feasibility set of decision profiles is given as

$$\begin{aligned} \Omega = \{x \in \mathbb{R}^{2n} | \sum_{j=1}^{4} x_j^1 = \sum_{j=1}^{4} r_j, G_j(x_j) \leq 0, \\ L_{j,2} x_j = \mathbf{0}_{2n_j}, \forall j \in \{1, \cdots, 4\}\}, \end{aligned} \tag{31}$$

where $n = 16$, $n_1 = 4$, $n_2 = 3$, $n_3 = 5$, $n_4 = 4$, $x_j^i = [(x_j^i)^1, (x_j^i)^2]^T \in \mathbb{R}^2, i \in \{1, \cdots, n_j\}$, $F_j(x_j, x_{-j}) = \sum_{i=1}^{n_j} f_{i,j}(x_j^i, S_j^i(x_{-j}))$, $G_j(x_j) = \sum_{i=1}^{n_j} g_j^i(x_j^i)$, $r = col(r_1, r_2, r_3, r_4) = [-3, 1, 5, 3, 2, -3, -2, -4]^T$, $\sum_{j=1}^{4} r_j = [2, -3]^T$, $j \in \{1, 2, 3, 4\}$. The local payoff function $f_i(x_i)$ and local nonsmooth inequality constraint $g_j^i(x_j^i)$ for agent $i \in \{1, \cdots, n_j\}$ in the cluster j are defined as

$$\begin{aligned} & f_{i,j}^1(x_j^i, S_j^i(x_{-j})) = \begin{cases} 2\|x_j^1 - p_j^1\|^2 + \sum_{k \in \mathcal{N}_0^j} (x_j^1)^T x_k^1, & \text{if } i = 1, \\ 2\|x_j^i - p_j^i\|^2, & \text{otherwise}, \end{cases} \\ & f_{i,j}^2(x_j^i, S_j^i(x_{-j})) = \begin{cases} 0, & \text{if } x_j^i \in \Omega_j^i \\ \infty, & \text{if } x_j^i \notin \Omega_j^i \end{cases}, \\ & g_j^i(x_j^i) = \|x_j^i - q_j^i\|_1 - c_j^i, \end{aligned} \tag{32}$$

where $k \in \{1, \cdots, 16\}$, $p_j^i = [k - 8, k - 8]^T$, $q_j^i = [-0.5, k - 10]^T$, $c = col(c_1^1, \cdots, c_4^4) = [26.1,\ 25.5,\ 26.1,\ 24.5,\ 24.2,\ 25.9,\ 26.5,\ 27.5,\ 24.1,\ 25.1,\ 27.1,\ 26.4,\ 0.8,\ 20.3,\ 20.7,\ 20.8]^T$, and $\Omega_j^i = \{\delta \in \mathbb{R}^2 | \|\delta - x_j^i(0)\|^2 \leq 49\}$. For each agent i within the cluster j, $f_i^0(x_i)$, $f_i^1(x_i)$ and $g_i(x_i)$ represents the quadratic cost function, the indicator function of $x_j^i \in \Omega_j^i$, and the l_1 penalty for an anchor q_j^i, respectively.

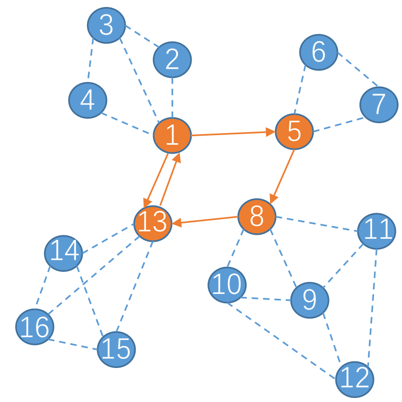

Fig. 1. The whole communication topology of the multi-agent system

The communication topology $\mathcal{G}$ of the multi-agent system, combined by the inter-cluster graph and inner-cluster graphs involved in problem (4), is shown in Fig.1. There are four clusters in this game, which are formulated as $J_1 = \{1, 2, 3, 4\}$, $J_2 = \{5, 6, 7\}$, $J_3 = \{8, 9, 10, 11, 12\}$, and $J_4 = \{13, 14, 15, 16\}$. The leaders of four clusters are agent 1, agent 5, agent 8, and agent 13, respectively. The blue dashed lines denote edges of inner-cluster undirected graphs, while the orange solid lines denote edges of the inter-cluster directed graph. The initial positions of the agents in clusters $\{J_1, J_2, J_3, J_4\}$ can be randomly located within areas $R_1 = \{\delta \in \mathbb{R}^2 | \|\delta - [-9, 6.5]^T\|^2 \leq 32\}$, $R_2 = \{\delta \in \mathbb{R}^2 | \|\delta - [4, 6]^T\|^2 \leq 18\}$, $R_3 = \{\delta \in \mathbb{R}^2 | \|\delta - [5, -5.5]^T\|^2 \leq 32\}$, and $R_4 = \{\delta \in \mathbb{R}^2 | \|\delta - [-7, -4.5]^T\|^2 \leq 20\}$, respectively. The initial values of Lagrangian multipliers μ, v, ω, and auxiliary variables y, η are set to zeros.

The final iteration step is 3000. The running time step size is 0.1 s. The real running time is 330.7 s. The error of the sequence $x(k)$ is defined as $ER(k) = \|x(k) - x(k-1)\|$ for $k \in \{1, \cdots, 3000\}$. The final error is $ER(3000) = 0.0010$. Motions of multi-agent system with algorithm (9) are presented in Fig.2, showing that players in the same cluster achieve consensus. Fig.3 gives trajectories of $\sum_{j=1}^{4} (x_j^1)^k(t)$ for $k \in \{1, 2\}$, $j \in \{1, 2, 3, 4\}$, indicating that the inter-cluster resource allocation constraint is satisfied. Fig.4 presents trajectories of $G_j(x_j(t))$ for $j \in \{1, 2, 3, 4\}$, showing that the coupled nonsmooth inequality constraints for clusters are satisfied. From Fig.2-Fig.4, it is evident that all agents achieve the GNE of this multi-cluster game, minimizing the global payoff function and satisfying nonsmooth inequality constraints and the resource allocation constraint.

As a comparative result, a distributed nonsmooth algorithm, which directly employs the subgradients of nonsmooth functions like classic nonsmooth algorithms [21]–[23] does, is

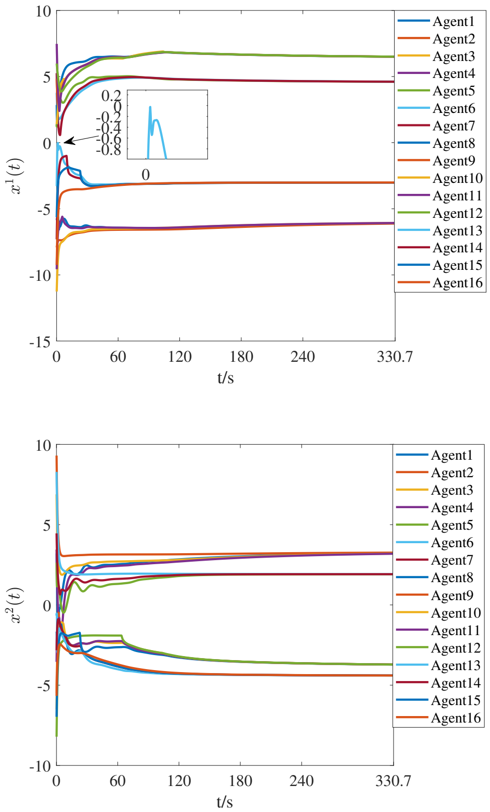

Fig. 2. Trajectories of $x_j^i(t)$ with $i \in \{1, \cdots, n_j\}$, $j \in \{1, 2, 3, 4\}$ under algorithm (9)

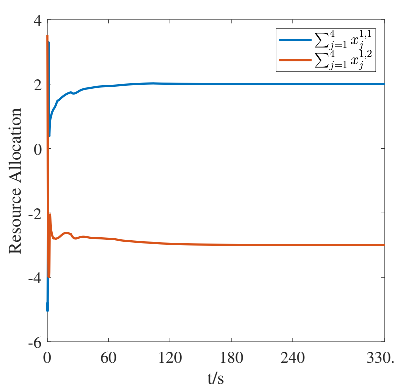

Fig. 3. Trajectories of $\sum_{j=1}^{4}(x_j^1)^k(t)$ with $k \in \{1, 2\}$, $j \in \{1, 2, 3, 4\}$ under algorithm (9)

introduced as follows:

$$
\begin{aligned}
\dot{x} \in & \mathbf{P}_\Omega\big[x - \alpha_1 \nabla F^1(x) - \omega^T \partial g(x) - \alpha_3 \mathbf{L}_{n,q}\mu \\
& - \alpha_3 \mathbf{L}_{n,q} x + \bar{v}\big] - x, \\
\dot{\omega} = & \mathbf{P}_{\mathbb{R}_+^n}[\omega + \alpha_2 g(x) - \alpha_6 \mathbf{L}\omega - \alpha_6 \mathbf{L}\xi] - \omega, \\
\dot{\mu} = & \alpha_3 \mathbf{L}_{n,q} x, \dot{\xi} = \alpha_6 \mathbf{L}\omega, \\
\dot{v} = & - H^{-1}(x^1 + \dot{x}^1 - r) - \alpha_5 L_0 v - \eta, \\
\dot{\eta} = & \alpha_5 L_0 v, \eta(0) = \mathbf{0}_{mq}.
\end{aligned} \tag{33}
$$

The final iteration step is 3000. The running time step size is 0.1 s. The real running time is 355.7 s. The final error is $ER(3000) = 0.0010$. The trajectories of x steered by algorithm (33) are presented in Fig.5. From Fig. 5, we can observe that the trajectory of x_{13} has vibrations from 0s to 10s, which should be avoided in the distributed GNE seeking process implemented by physical systems.

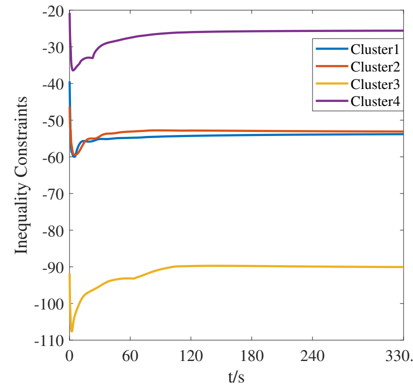

Fig. 4. Trajectories of $G_j(x_j(t))$ with $j \in \{1, 2, 3, 4\}$ under algorithm (9)

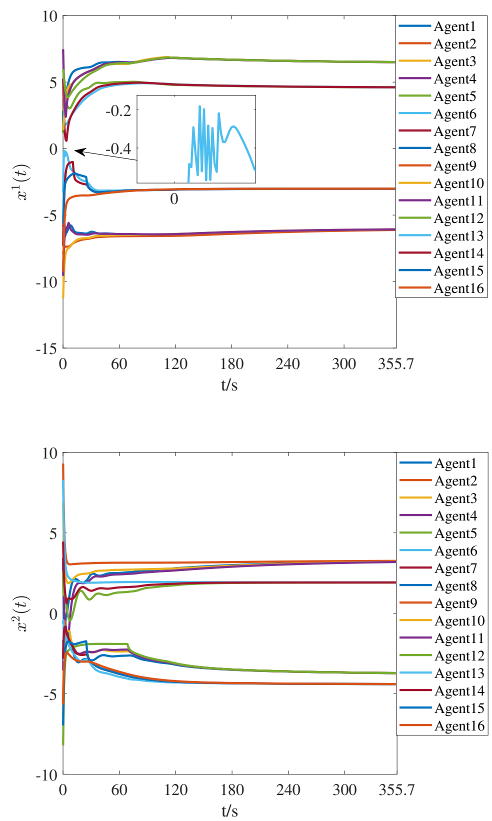

Fig. 5. Trajectories of $x_j^i(t)$ with $i \in \{1, \cdots, n_j\}$, $j \in \{1, 2, 3, 4\}$ under algorithm (33)

VII. Conclusion

A GNE seeking strategy for a class of nonsmooth constrained multi-cluster noncooperative games is investigated in this paper. Each player in this game deals with two nonsmooth functions: a nonsmooth payoff function and a nonsmooth function in the coupled inequality constraint, respectively. Players in the same cluster should cooperate to satisfy a nonsmooth coupled inequality constraint. A distributed GNE seeking algorithm is presented under the directed inter-cluster graph. Two proximal operators are involved in this algorithm to tackle these two nonsmooth functions mentioned above separately. In the future work, the switching directed graph, the order of proximal operators, and nonconvex payoff functions will be considered in nonsmooth multi-cluster games.

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