An Under-Approximation for the Robust Uncertain Two-Level Cooperative Set Covering Problem

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Outline

- Introduction
- Model Formulation
- Computational Experiments

Introduction

Set Covering Problem

- Minimum the total cost of cover
- Selection of a subset of location sites
- Cover a set of demand nodes
- NP-complete
- Application in facility location problems
  - locate emergency service facilities
  - node deployment

$$\min \sum_{j\in\mathcal{J}} c_j x_j$$

$$\text{s.t.} \sum_{j\in\mathcal{J}} a_{ij} x_j \geq 1 \qquad \forall i \in \mathcal{I}$$

$$x_j \in \{0,1\} \qquad \forall j \in \mathcal{J}$$

- a set $\mathcal{I}$ of *m* demand nodes
- a set $\mathcal{J}$ of n potential facility location sites
- 0-1 matrix A indicates whether a location *j* is able to cover a demand node *i*

Literature Review

- Probabilistic set covering problem

Beraldi and Ruszczynski (2002)

- Two-level facility location problem

Aardal et al. (1996)

- Joint resource allocation problem

Xin et al. (2018), Wang et al. (2019), Xu et al. (2020)

- Robust set covering problem

Pereira, and Averbakh (2013), Lutter et al. (2017)

The works related to SCP are **linear** programming problems, only joint resource allocation problem considers **nonlinear** functions in the objective functions

Literature Review

- Commonly used methods
  - Branch-and-bound: Beraldi and Ruszczynski (2002)
  - Branch-and-cut: Pereira, and Averbakh (2013)
  - Cutting plane approach: Lutter et al. (2017)
  - Benders decomposition: Pereira, and Averbakh (2013)
  - Heuristics & Meta-heuristics
    - Greedy heuristic: Chvatal (1979)
    - Marginal-return-based constructive heuristic: Xin et al. (2018)
    - Genetic algorithm: Pereira, and Averbakh (2013)
    - Memetic algorithm: Wang et al. (2019)
    - MOEA/D: Xu et al. (2020)

Heuristics and meta-heuristics can solve the **nonlinear** programming, but **none** of the works consider **approximating** the problem with **linear** programming model.

Motivation

- Multi-platforms location
- Perform cooperation tasks
- Probabilistic covering with uncertainty
- Unlike past works use heuristics & meta-heuristics

We introduce a novel set covering problem with **cooperation tasks under uncertainty** as an integer **nonlinear** programming and considers **linear approximation technique** to solve the problem with **near-optimal** solutions in **real-time**.

Paper Contribution

- A compact mixed-integer linear programming formulation is proposed by utilizing robust optimization and constraint relaxation.
- The proposed formulation is analyzed on a large set of test cases with 10125 different instances.
- A majority of the under-approximate solutions are proven to be optimal while few of them slightly violate the constraints and provide an efficient lower bound.

Model Formulation

Two-Level Cooperative Set Covering Problem (TLCSCP)

$$\begin{aligned} \min \quad & \sum_{j\in\mathcal{J}} c_j^1 y_j + \sum_{k\in\mathcal{K}} c_k^2 z_k \\ \text{s.t.} \quad & \left(\sum_{j\in\mathcal{J}} a_{ij} y_j\right) \cdot \left(\sum_{k\in\mathcal{K}} b_{ik} z_k\right) \geq 1 \\ & y_j \in \{0,1\} \\ & z_k \in \{0,1\} \end{aligned}$$

a_{ij} and b_{ij} are binary

$$\begin{aligned} \min \quad & \sum_{j\in\mathcal{J}} c_j^1 y_j + \sum_{k\in\mathcal{K}} c_k^2 z_k \\ \text{s.t.} \quad & \sum_{j\in\mathcal{J}} a_{ij} y_j \geq 1 \\ & \sum_{k\in\mathcal{K}} b_{ik} z_k \geq 1 \\ & y_j \in \{0,1\} \\ & z_k \in \{0,1\} \end{aligned}$$

Generalized Uncertain Two-Level Cooperative Set Covering Problem (GUTLCSCP)

$$\begin{aligned} \min \quad & \sum_{j \in \mathcal{J}} c_j^1 y_j + \sum_{k \in \mathcal{K}} c_k^2 z_k \\ \text{s.t.} \quad & P\left(\sum_{j \in \mathcal{J}} a_{ij} y_j \geq 1, \sum_{k \in \mathcal{K}} b_{ik} z_k \geq 1\right) \geq \alpha \\ & y_j \in \{0,1\} \\ & z_k \in \{0,1\} \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} \min \quad & \sum_{j \in \mathcal{J}} c_j^1 y_j + \sum_{k \in \mathcal{K}} c_k^2 z_k \\ \text{s.t.} \quad & \left(1 - \prod_{j \in \mathcal{J}} p_{ij}^{y_j}\right) \cdot \left(1 - \prod_{k \in \mathcal{K}} q_{ik}^{z_k}\right) \geq \alpha \\ & y_j \in \{0,1\} \\ & z_k \in \{0,1\} \end{aligned}$$

- Probabilistic covering

$$P\left(\sum_{j \in \mathcal{C}^1} a_{ij} \geq 1\right) = 1 - \prod_{j \in \mathcal{C}^1} p_{ij}, \quad P\left(\sum_{k \in \mathcal{C}^2} b_{ik} \geq 1\right) = 1 - \prod_{k \in \mathcal{C}^2} q_{ik}.$$

Linear approximation reformulation

$$\begin{cases} m_i = \prod_{j\in\mathcal{J}} p_{ij}^{y_j} \\ n_i = \prod_{k\in\mathcal{K}} q_{ik}^{z_k} \\ (1-m_i)(1-n_i) \geq \alpha \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \ln(m_i) = \sum_{j\in\mathcal{J}} \ln(p_{ij}) y_j \\ \ln(n_i) = \sum_{k\in\mathcal{K}} \ln(q_{ik}) z_k, \\ (1-m_i)(1-n_i) \geq \alpha \end{cases}$$

$$\begin{aligned} \min \quad & \sum_{j\in\mathcal{J}} c_j^1 y_j + \sum_{k\in\mathcal{K}} c_k^2 z_k && \\ \text{s.t.} \quad & \ln(m_i) = \sum_{j\in\mathcal{J}} \ln(p_{ij}) y_j && \forall i \in \mathcal{I} \quad (14) \\ & \ln(n_i) = \sum_{k\in\mathcal{K}} \ln(q_{ik}) z_k && \forall i \in \mathcal{I} \quad (15) \\ & (1-m_i)(1-n_i) \geq \alpha && \forall i \in \mathcal{I} \quad (16) \\ & y_j \in \{0,1\} && \forall j \in \mathcal{J} \\ & z_k \in \{0,1\} && \forall k \in \mathcal{K} \\ & 0 \leq m_i \leq 1 && \forall i \in \mathcal{I} \quad (17) \\ & 0 \leq n_i \leq 1 && \forall i \in \mathcal{I}. \quad (18) \end{aligned}$$

Linear approximation reformulation

How to approximate the nonlinear constraints (16)?

$$(1-m_i)(1-n_i) \geq \alpha$$

1. Approximate the two terms in the left-hand-side by two linear constraints, respectively

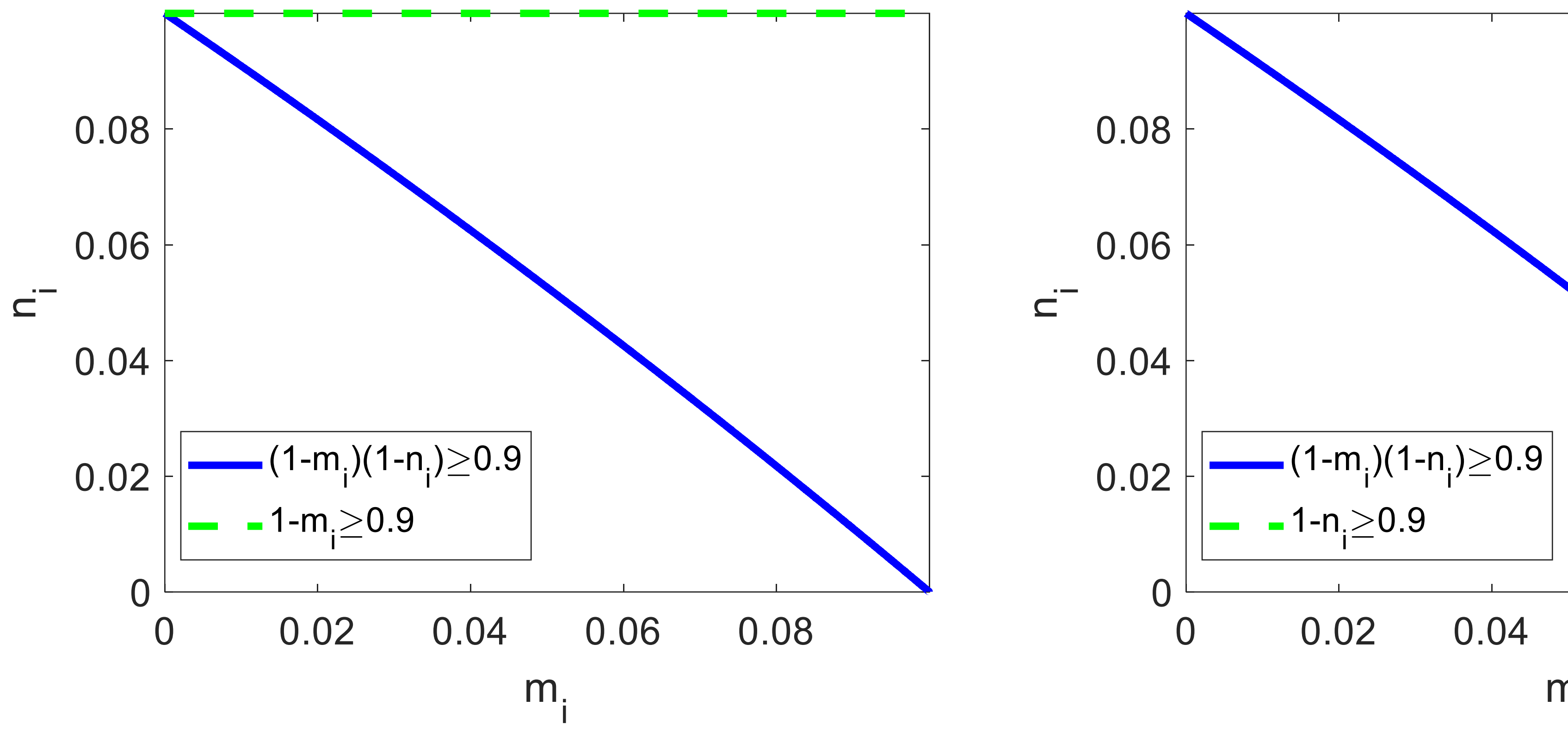
$$\begin{cases} 1-m_i \geq \alpha \\ 1-n_i \geq \alpha \end{cases} \Longleftrightarrow \begin{cases} \sum_{j\in\mathcal{J}} \ln(p_{ij})y_j \leq \ln(1-\alpha) \\ \sum_{k\in\mathcal{K}} \ln(q_{ik})z_k \leq \ln(1-\alpha) \end{cases}$$

2. Approximate the two terms in the left-hand-side together by one linear constraint tangent to (16)

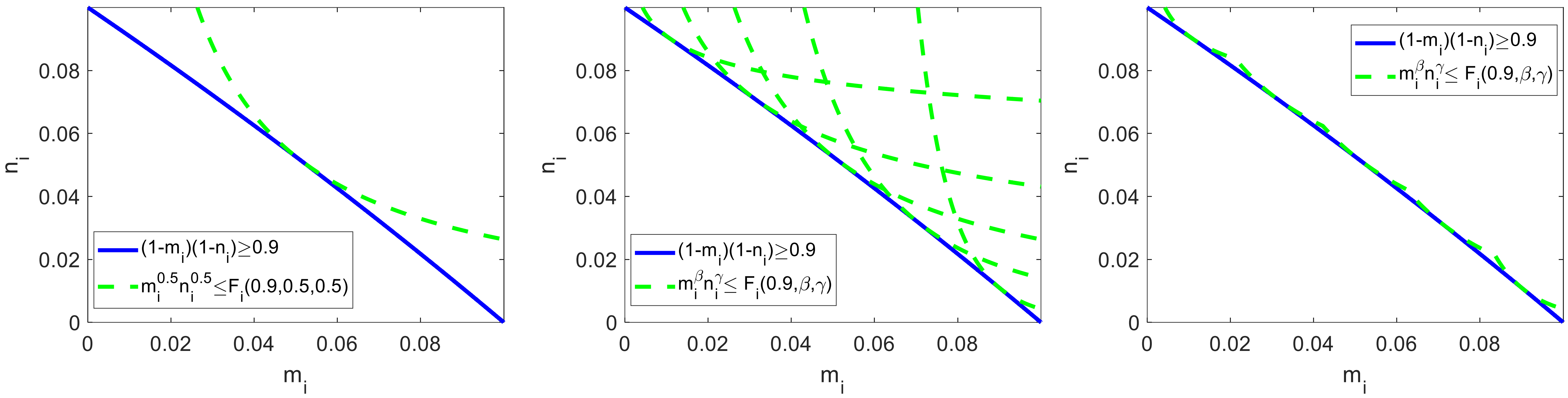
$$\beta \ln(m_i) + \gamma \ln(n_i) \leq \ln(F_i(\alpha,\beta,\gamma)) \Longleftrightarrow m_i^{\beta} n_i^{\gamma} \leq F_i(\alpha,\beta,\gamma)$$

substitute by m_i: the equality equation (16) $m_i = 1-\alpha/(1-n_i)$ and obtain the parameters of F_i by calculating the tangent point.

$$1 - m_i \geq \alpha \text{ and } 1 - n_i \geq \alpha\,, \quad \alpha = 0.9$$



$$\beta \ln(m_i) + \gamma \ln(n_i) \leq \ln(F_i(\alpha, \beta, \gamma)) \quad \alpha = 0.9 \quad \beta + \gamma = 1$$



(a) $\beta = 0.5,\ \gamma = 0.5$

(b) multiple constraints with different β/γ

(c) multiple constraints with different β/γ after combination

Linear approximation formulation of the GUTLCSCP (GUTLCSCP-LA)

$$
\begin{aligned}
\min \quad & \sum_{j\in\mathcal{J}} c_j^1 y_j + \sum_{k\in\mathcal{K}} c_k^2 z_k \\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} \ln(p_{ij}) y_j \le \ln(1-\alpha) && \forall i \in \mathcal{I} \\
& \sum_{k\in\mathcal{K}} \ln(q_{ik}) z_k \le \ln(1-\alpha) && \forall i \in \mathcal{I} \\
& \beta \sum_{j\in\mathcal{J}} \ln(p_{ij}) y_j + \gamma \sum_{k\in\mathcal{K}} \ln(q_{ik}) z_k \le \ln\left[\left(1-\frac{\alpha}{1-\delta}\right)^{\beta} n_i^{\gamma}\right] && \forall i \in \mathcal{I} \\
& y_j \in \{0,1\} && \forall j \in \mathcal{J} \\
& z_k \in \{0,1\} && \forall k \in \mathcal{K},
\end{aligned}
$$

where $\delta = \frac{2\gamma+\alpha\beta-\alpha\gamma-\sqrt{\alpha(4\beta\gamma+\alpha\beta^2+\alpha\gamma^2-2\alpha\beta\gamma)}}{2\gamma}$. $\beta,\ \gamma \in [0,1]$ are constants or vectors with $\beta+\gamma=1$.

Modeling the Robust Uncertain Two-Level Cooperative Set Covering Problem

- Based on GUTLCSCP
- The probabilities p_{ij} and q_{ik} are uncertain
  - within the interval $[\bar{p}_{ij}, \bar{p}_{ij}+\hat{p}_{ij}] \subseteq [0,1]$ and $[\bar{q}_{ik}, \bar{q}_{ik}+\hat{q}_{ik}] \subseteq [0,1]$
  - controlled by the budget of uncertainty Γ
- Two Γ–scenario uncertainty sets

$$\mathscr{U}_1^{\Gamma_i} := \left\{ p_{i:} | \forall j \in \mathcal{J} : p_{ij} \in [\bar{p}_{ij}, \bar{p}_{ij}+\hat{p}_{ij}], \sum_{j\in\mathcal{J}} \frac{p_{ij}-\bar{p}_{ij}}{\hat{p}_{ij}} \le \Gamma_i \right\}$$

$$\mathscr{U}_2^{\Gamma_i} := \left\{ q_{i:} | \forall k \in \mathcal{K} : q_{ik} \in [\bar{q}_{ik}, \bar{q}_{ik}+\hat{q}_{ik}], \sum_{k\in\mathcal{K}} \frac{q_{ik}-\bar{q}_{ik}}{\hat{q}_{ik}} \le \Gamma_i \right\},$$

for all $i \in \mathcal{I}$, where $p_{i:} := (p_{ij})_{j\in\mathcal{J}}$, $q_{i:} := (q_{ik})_{k\in\mathcal{K}}$.

Robust Uncertain Two-Level Cooperative Set Covering Problem (RUTLCSCP)

$$\min \sum_{j\in\mathcal{J}} c_j^1 y_j + \sum_{k\in\mathcal{K}} c_j^2 z_k$$

Γ-robust two-level-cooperative α-cover →

$$\text{s.t.} \quad \left[1-\beta_i^1(y,\Gamma_i)\right]\cdot\left[1-\beta_i^2(z,\Gamma_i)\right] \ge \alpha \qquad \forall i\in\mathcal{I}$$

$$y_j \in \{0,1\} \qquad \forall j\in\mathcal{J}$$

$$z_k \in \{0,1\} \qquad \forall k\in\mathcal{K}$$

$$\beta_i^1(y,\Gamma_i) := \max_{\{\mathcal{U}_1\subseteq\mathcal{C}^1(y):|\mathcal{U}_1|\le\Gamma_i\}} \left\{ \prod_{j\in\mathcal{U}_1} (\bar{p}_{ij}+\hat{p}_{ij})^{y_j} \cdot \prod_{j\in\mathcal{J}\setminus\mathcal{U}_1} \bar{p}_{ij}^{y_j} \right\}$$

$$\beta_i^2(z,\Gamma_i) := \max_{\{\mathcal{U}_2\subseteq\mathcal{C}^2(z):|\mathcal{U}_2|\le\Gamma_i\}} \left\{ \prod_{k\in\mathcal{U}_2} (\bar{q}_{ik}+\hat{q}_{ik})^{z_k} \cdot \prod_{k\in\mathcal{K}\setminus\mathcal{U}_2} \bar{q}_{ik}^{z_k} \right\}$$

Robust counterpart (RC) of RUTLCSCP with linear approximation (RUTLCSCP-LA-RC)

$$\min \sum_{j\in\mathcal{J}} c_j^1 y_j + \sum_{k\in\mathcal{K}} c_j^2 z_k$$

$$\text{s.t.} \sum_{j\in\mathcal{J}} \ln(\bar{p}_{ij}) y_j + \sum_{j\in\mathcal{J}} \zeta_{ij}^1 + \Gamma_i \eta_i^1 \le \ln(1-\alpha)$$

$$\sum_{k\in\mathcal{K}} \ln(\bar{q}_{ik}) z_k + \sum_{k\in\mathcal{K}} \zeta_{ik}^2 + \Gamma_i \eta_i^2 \le \ln(1-\alpha)$$

$$\beta\left[\sum_{j\in\mathcal{J}} \ln(\bar{p}_{ij}) y_j + \sum_{j\in\mathcal{J}} \zeta_{ij}^1 + \Gamma_i \eta_i^1\right] + \gamma\left[\sum_{k\in\mathcal{K}} \ln(\bar{q}_{ik}) z_k + \sum_{k\in\mathcal{K}} \zeta_{ik}^2 + \Gamma_i \eta_i^2\right]$$

$$\le \ln\left[\left(1-\frac{\alpha}{1-\delta}\right)^{\beta} n_i^{\gamma}\right]$$

$$\zeta_{ij}^1 + \eta_i^1 \ge \left(\ln(\bar{p}_{ij}+\hat{p}_{ij}) - \ln(\hat{p}_{ij})\right) y_j$$

$$\zeta_{ik}^2 + \eta_i^2 \ge \left(\ln(\bar{q}_{ik}+\hat{q}_{ik}) - \ln(\hat{q}_{ik})\right) z_k$$

$$\zeta_{ij}^1 \ge 0$$

$$\zeta_{ik}^2 \ge 0$$

$$\eta_i^1 \ge 0$$

$$\eta_i^2 \ge 0$$

$$y_j \in \{0,1\}$$

$$z_k \in \{0,1\}$$

transformed to a mixed-integer linear programming (MILP) problem by the strong duality theorem and linear approximation

Computational Experiments

Computational Experiments

- Fixed cost
- Uncertain probability
- $\alpha \in \{0.8, 0.85, 0.9\}$
- $\Gamma \in \{0, \ldots, |\mathcal{I}|\}$
- Solved by Cplex

THE TEST-CASE FOR RUTLCSCP

Instance	$(\lvert\mathcal{I}\rvert, \lvert\mathcal{J}\rvert, \lvert\mathcal{K}\rvert)$	$(yr/km, zr/km)$	$(A_x/km, A_y/km)$
P1.1–P1.5	$(20, 20, 20)$	$(10, 5)$	$(25, 25)$
P2.1–P2.5	$(25, 25, 25)$	$(10, 5)$	$(25, 25)$
P3.1–P3.5	$(30, 30, 30)$	$(10, 5)$	$(25, 25)$
P4.1–P4.5	$(40, 40, 40)$	$(14, 7)$	$(50, 50)$
P5.1–P5.5	$(50, 50, 50)$	$(14, 7)$	$(50, 50)$
P6.1–P6.5	$(60, 60, 60)$	$(14, 7)$	$(50, 50)$
P7.1–P7.5	$(80, 80, 80)$	$(20, 10)$	$(100, 100)$
P8.1–P8.5	$(100, 100, 100)$	$(20, 10)$	$(100, 100)$
P9.1–P9.5	$(120, 120, 120)$	$(20, 10)$	$(100, 100)$
P10.1–P10.5	$(140, 140, 140)$	$(20, 10)$	$(100, 100)$

Each demand node serves as a candidate location site for y-facility and z-facility

number of demand nodes (candidate locations)

covering ranges

positions of the demand nodes (candidate locations)

Computational Experiments

TABLE II

COMPUTATIONAL RESULTS FOR RUTLCSCP

Instance	$\alpha = 0.8^*$			$\alpha = 0.85^*$			$\alpha = 0.9$			
	Opt. (%)	Time	CV (%)	Opt. (%)	Time	CV (%)	Opt. (%)	Time	CV (%)	Degree of feasibility (%)
P1.1–P1.5	100.00	0.14	0.00	100.00	0.17	0.00	100.00	0.10	0.00	61.90
P2.1–P2.5	100.00	0.21	0.00	100.00	0.25	0.00	98.75	0.20	0.05	61.54
P3.1–P3.5	100.00	0.24	0.00	100.00	0.33	0.00	99.20	0.37	0.03	80.65
P4.1–P4.5	100.00	0.32	0.00	100.00	0.48	0.00	100.00	0.30	0.00	21.95
P5.1–P5.5	100.00	0.56	0.00	100.00	0.76	0.00	100.00	0.44	0.00	41.18
P6.1–P6.5	100.00	1.03	0.00	100.00	1.01	0.00	100.00	0.44	0.00	21.31
P7.1–P7.5	80.25	1.48	0.25	100.00	1.60	0.00	100.00	0.74	0.00	1.23
P8.1–P8.5	100.00	3.00	0.00	80.00	3.76	0.20	96.10	2.81	0.04	40.59
P9.1–P9.5	100.00	4.59	0.00	100.00	5.98	0.00	99.18	4.58	0.01	40.50
P10.1–P10.5	100.00	7.29	0.00	80.14	8.33	0.14	100.00	5.68	0.00	20.57

* Degree of feasibility are 100%.

Computational Experiments

- total constraint violations ϕ
- proportion of violations with total nonlinear constraints $\#$

In summary, a set of 10125 instances are generated and solved with good quality and acceptable time. Up to 74.10% (7502 instances) are solved to optimality, 3.29% (333 instances) are under-approximation, and 22.62% (2290 instances) are with no solution.

TABLE III

INSTANCE TYPES WITH CONSTRAINT VIOLATION

Instance	α	Γ	Obj.	ϕ	#
P2.2	0.9	0	463.94	7.40E-06	1/20
P3.1	0.9	0	309.00	1.29E-05	1/25
P7.1	0.8	1+	1464.12	3.79E-04	1/80
P8.1	0.9	0	1258.93	1.13E-05	1/100
P8.2	0.85	0	1514.56	4.21E-04	1/100
P8.2	0.9	0	1603.00	8.01E-06	1/100
P8.3	0.85	1+	1444.47	2.51E-04‡	1/100
P8.3	0.9	0	1409.12	1.93E-04	1/100
P8.3	0.9	2+	1942.41†	2.12E-04‡	1/100
P9.2	0.9	0	1501.84	3.91E-04	1/120
P9.3	0.9	0	1312.91	3.97E-06	1/120
P10.3	0.85	1+	1408.94	3.44E-04	1/140

† The objective values are still varying as different Γ.
‡ The total constraint violations are varying with different Γ.

Concluding Remarks

- consider two types of facilities
- under-approximation with a larger feasible region
- computation time up to 10 seconds

Future Research

- over-approximation with a smaller feasible region
- cooperation with more than two types of facilities
- more real-world applications

Thank you for your attention!

Q&A