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Distributed Nonsmooth Robust Resource Allocation with Cardinality Constrained Uncertainty

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Background

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Background

Most of the existing works about distributed resource allocation problem have the assumption that the resource allocation condition is deterministic.

Nonsmooth optimization problem is increasingly popular due to its important role in a lot of signal processing, statistical inference and machine learning problems.

Background

Our Problem:

The nonsmooth robust resource allocation problem we investigate here is with cardinality constrained uncertain parameters, which decrease the conservatism of the problem using polyhedral uncertain parameters.

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Problem Formulation

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Problem Formulation

Distributed nonsmooth uncertain resource allocation problem:

$$\min_{x_i \in \Omega_i} \quad f(x) = \sum_{i=1}^{n} f_i(x_i)$$

$$\text{s.t.} \sum_{i=1}^{n} \bar{a}_{ij}^l x_i^l \leq b_j^l, \quad \forall \bar{a}_{ij}^l \in \mathcal{U}_{ij}^l$$

Where

$$\mathcal{U}_{ij}^l = \{\bar{a}_{ij}^l | \bar{a}_{ij}^l \in [a_{ij}^l - \hat{a}_{ij}^l, a_{ij}^l + \hat{a}_{ij}^l],$$

$$\sum_{i,l} \left| \frac{\bar{a}_{ij}^l - a_{ij}^l}{\hat{a}_{ij}^l} \right| \leq \gamma_j, \forall i, j, l\}$$

Problem Formulation

Corresponding robust optimization problem:

$$\min_{x_i \in \Omega_i} \quad f(x) = \sum_{i=1}^{n} f^i(x_i)$$

$$\text{s.t.} \sum_{i=1}^{n} a_{ij}^l x_i^l + \max_{S_j^l \in J_j^l : |S_j^l| = \gamma_j} \sum_{i \in S_j^l} \hat{a}_{ij}^l x_i^l \leq b_j^l,$$

$$j \in \{1, \cdots, m\}, l \in \{1, \cdots, q\}$$

Corresponding dual problem:

$$\min_{x_i \in \Omega_i} \quad f(x) = \sum_{i=1}^{n} f^i(x_i)$$

$$\text{s.t.} \sum_{i=1}^{n} [A_{ij} x_i + \frac{1}{n} \gamma_j z_{ij} + w_{ij}] \leq \sum_{i=1}^{n} b_{ij},$$

$$\hat{A}_{ij} x_i \leq z_{ij} + w_{ij}, L_{mnq} Z = \mathbf{0}_{mnq},$$

$$z_{ij} \geq \mathbf{0}_q, w_{ij} \geq \mathbf{0}_q,$$

$$i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$$

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Algorithm Design

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Algorithm Design

Algorithm Form:

$$
\begin{cases}
\dot{\bar{x}}_i \in -\bar{x}_i + x_i - \partial f_i(x_i) - \sum_{j=1}^{m} A_{ij}\lambda_{ij}^1 - \sum_{j=1}^{m} \hat{A}_{ij}\lambda_{ij}^2 \\
\dot{\bar{z}}_{ij} = -\bar{z}_{ij} + z_{ij} - \frac{1}{n}\gamma_j \lambda_{ij}^1 + \lambda_{ij}^2 - \sum_{k\in\mathcal{N}_i} \alpha_k(\mu_{ij} - \mu_{kj}) \\
\dot{\bar{w}}_{ij} = -\bar{w}_{ij} + w_{ij} - \lambda_{ij}^1 + \lambda_{ij}^2 \\
\dot{\mu}_{ij} = \sum_{k\in\mathcal{N}_i} \alpha_{ik}(z_{ij} - z_{kj}) \\
\dot{\bar{\lambda}}_{ij}^1 = -\bar{\lambda}_{ij}^1 + \lambda_{ij}^1 + [A_{ij}x_i + \frac{1}{n}\gamma_j z_{ij} + w_{ij} - b_{ij}] \\
\qquad + \sum_{k\in\mathcal{N}_i} \alpha_{ik}(y_{ij}^1 - y_{kj}^1) - \sum_{k\in\mathcal{N}_i} \alpha_{ik}(\lambda_{ij}^1 - \lambda_{kj}^1) \\
\dot{\bar{\lambda}}_{ij}^2 = -\bar{\lambda}_{ij}^2 + \lambda_{ij}^2 + [\hat{A}_{ij}x_i - z_{ij} - w_{ij}] \\
\qquad + \sum_{k\in\mathcal{N}_i} \alpha_{ik}(y_{ij}^2 - y_{kj}^2) - \sum_{k\in\mathcal{N}_i} \alpha_{ik}(\lambda_{ij}^2 - \lambda_{kj}^2) \\
\dot{y}_{ij}^1 = -\sum_{k\in\mathcal{N}_i} \alpha_{ik}(\lambda_{ij}^1 - \lambda_{kj}^1) \\
\dot{y}_{ij}^2 = -\sum_{k\in\mathcal{N}_i} \alpha_{ik}(\lambda_{ij}^2 - \lambda_{kj}^2) \\
x_i = P_{\Omega_i}[\bar{x}_i],\ z_{ij} = P_{\mathbb{R}_+^q}[\bar{z}_{ij}],\ w_{ij} = P_{\mathbb{R}_+^q}[\bar{w}_{ij}], \\
\lambda_{ij}^1 = P_{\mathbb{R}_+^q}[\bar{\lambda}_{ij}^1],\ \lambda_{ij}^2 = P_{\mathbb{R}_+^q}[\bar{\lambda}_{ij}^2]
\end{cases}
$$

Algorithm Design

Algorithm Form (Compact Form):

$$\dot{\Phi} \in \mathcal{F}(\Phi), x = P_{\Omega}[\bar{x}], Z = P_{\bar{\mathbb{R}}_{+}^{mnq}}[\bar{Z}],$$

$$W = P_{\bar{\mathbb{R}}_{+}^{mnq}}[\bar{W}], \Lambda^1 = P_{\bar{\mathbb{R}}_{+}^{mnq}}[\bar{\Lambda}^1], \Lambda^2 = P_{\bar{\mathbb{R}}_{+}^{mnq}}[\bar{\Lambda}^2]$$

where $\Phi = [\bar{x}^T, \bar{Z}^T, \bar{W}^T, U^T, (\bar{\Lambda}^1)^T, (\bar{\Lambda}^2)^T, (Y^1)^T, (Y^2)^T]^T$, $P_{\Omega}[\bar{x}] = [(P_{\Omega_1}[\bar{x}_1])^T, \cdots, (P_{\Omega_n}[\bar{x}_n])^T]^T$, $\bar{z}_i = [(\bar{z}_{i1})^T, \ldots, (\bar{z}_{im})^T]^T$, $\bar{Z} = [(\bar{z}_1)^T, \ldots, (\bar{z}_n)^T]^T$, $\bar{w}_i = [(\bar{w}_{i1})^T, \ldots, (\bar{w}_{im})^T]^T$, $\bar{W} = [(\bar{w}_1)^T, \ldots, (\bar{w}_n)^T]^T$, $\bar{\lambda}_i^g = [(\bar{\lambda}_{i1}^g)^T, \ldots, (\bar{\lambda}_{im}^g)^T]^T$, $\bar{\Lambda}^g = [(\bar{\lambda}_1^g)^T, \ldots, (\bar{\lambda}_n^g)^T]^T$, $y_i^g = [(y_{i1}^g)^T, \ldots, (y_{im}^g)^T]^T$, $Y^g = [(y_1^g)^T, \ldots, (y_n^g)^T]^T$, $g \in \{1, 2\}$.

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Convergence Analysis

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Convergence Analysis

Define the Lyapunov candidate:

$$\begin{aligned} V(\phi) =& V_1(\bar{x}) + V_2(\bar{Z}) + V_3(\bar{W}) + V_4(U) \\ &+ V_5(\bar{\Lambda}^1) + V_6(\bar{\Lambda}^2) + V_7(Y^1) + V_8(Y^2) \end{aligned}$$

Where

$$\begin{cases} V_1(\bar{x}) = \frac{1}{2}(\|\bar{x} - x^*\|^2 - \|\bar{x} - x\|^2) \\ V_2(\bar{Z}) = \frac{1}{2}(\|\bar{Z} - Z^*\|^2 - \|\bar{Z} - Z\|^2) \\ V_3(\bar{W}) = \frac{1}{2}(\|\bar{W} - W^*\|^2 - \|\bar{W} - W\|^2) \\ V_4(U) = \frac{1}{2}(\|U - U^*\|^2) \\ V_5(\bar{\Lambda}^1) = \frac{1}{2}(\|\bar{\Lambda}^1 - \Lambda^{1*}\|^2 - \|(\|\bar{\Lambda}^1 - \Lambda^1\|^2) \\ V_6(\bar{\Lambda}^2) = \frac{1}{2}(\|\bar{\Lambda}^2 - \Lambda^{2*}\|^2 - \|(\|\bar{\Lambda}^2 - \Lambda^2\|^2), \\ V_7(Y^1) = \frac{1}{2}(\|Y^1 - Y^{1*}\|^2) \\ V_8(Y^2) = \frac{1}{2}(\|Y^2 - Y^{2*}\|^2) \end{cases}$$

Convergence Analysis

Main Result:

Theorem 5.1. For Algorithm (8) with Assumption 3.1, we have that the results that

(i) the trajectory $(x, Z, W, \Lambda^1, \Lambda^2, \phi)$ is bounded;

(ii) $x(t)$ converges to the optimal solution to Problem (4).

Simulation

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Simulation

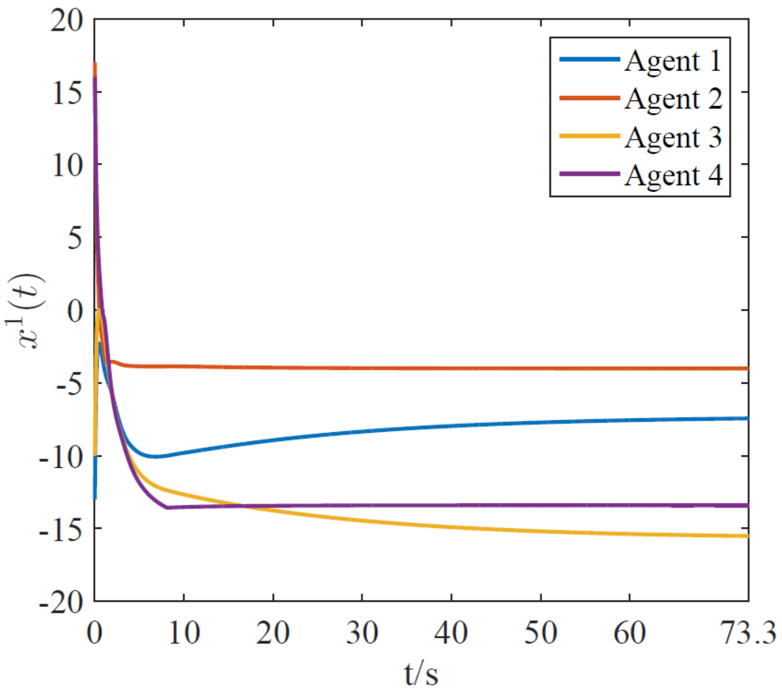
Consider the distributed robust optimization problem with four agents moving in a 2-D space with first-order dynamics as follows:

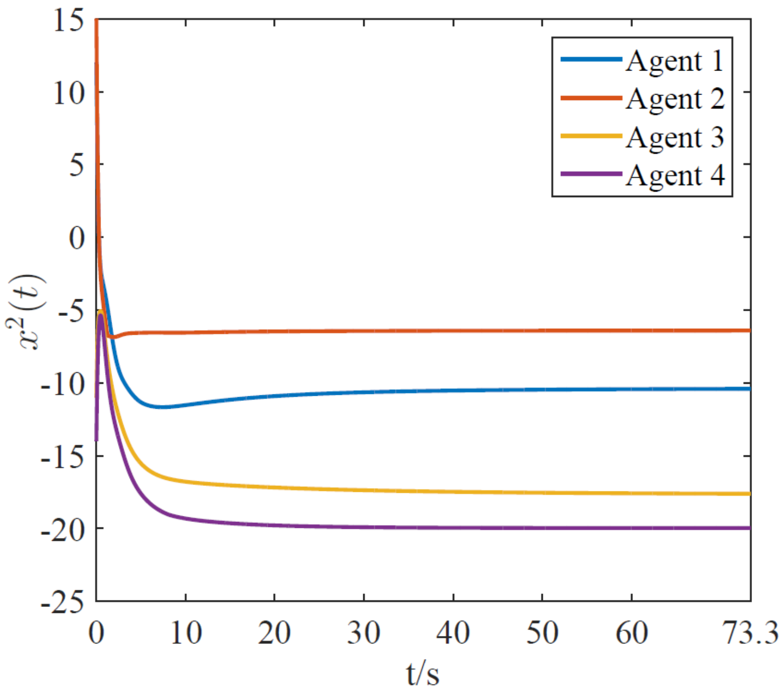
$$F(x) = \sum_{i=1}^{4} \|x_i - p_i\|_2^2 + |x|_1$$

where $p_i = [i, -i]^T$, $\| \cdot \|_2$ denotes the l_2 norm, $\Omega_i = \{\delta \in \mathbb{R}^2 | \|\delta - x_i(0)\|_2 \leq 30\}$, $m = 2$, $\gamma_1 = \gamma_2 = 2$, $A_{i1} = 0.1 \cdot i \cdot I_2$, $\hat{A}_{i1} = 0.1 \cdot (5 - i) \cdot I_2$, $A_{i2} = \hat{A}_{i1}$, $\hat{A}_{i2} = A_{i1}$, and $b_1^1 = [-15, -5]^T$, $b_1^2 = [-10, -4]^T$, $b_1^3 = [0, -6]^T$, $b_1^4 = [4, 0]^T$, $b_2^1 = [-5, -1]^T$, $b_2^2 = [-4, -3]^T$, $b_2^3 = [0, -2]^T$, $b_2^4 = [1, -5]^T$, $b_1 = [-21, -15]^T$, $b_2 = [-8, -11]^T$.

Simulation

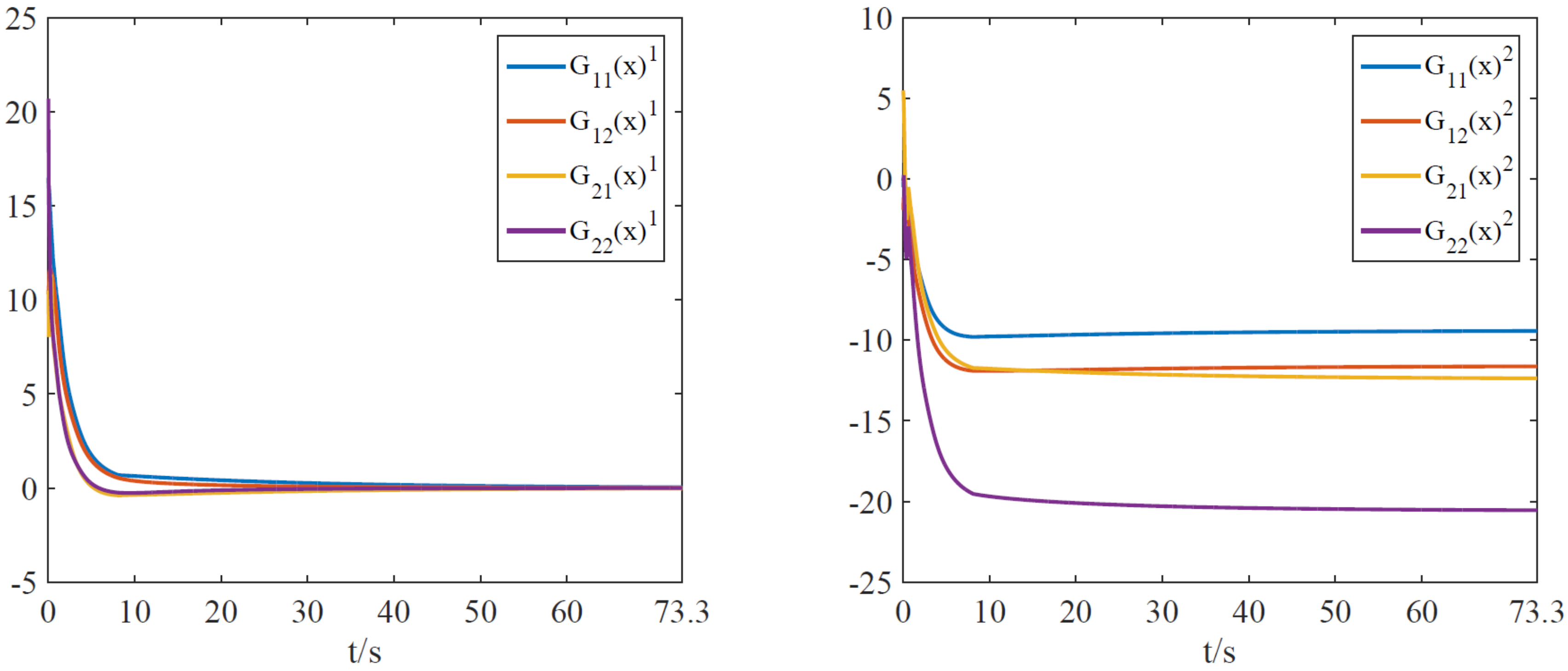
The trajectories of state x with the algorithm:





Simulation

The trajectories of resource allocation condition with the algorithm:



Conclusion

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Conclusion

1. In this paper, a distributed nonsmooth resource allocation problem with cardinality constrained uncertainty has been investigated.

2. A deterministic distributed robust resource allocation problem with linear optimization formulation has been derived under the framework of multi-agent system.

3. A distributed projection-based algorithm has been proposed to deal with this problem. Based on stability theory and differential inclusions, the proposed algorithm has been proved to reach the optimal solution and satisfy the resource allocation condition simultaneously.

Thank You!
Q&A